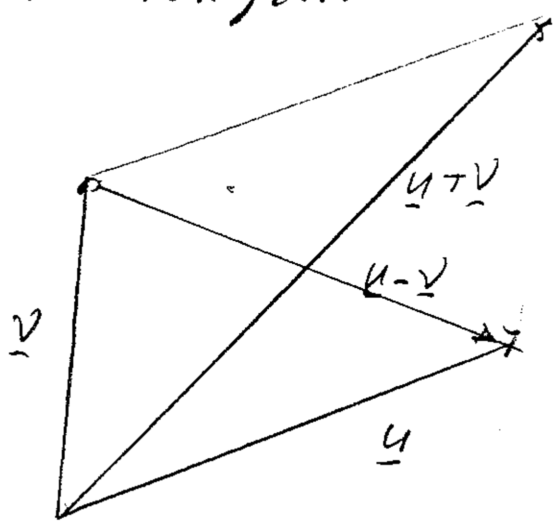
Midterm 1 Ma 53 Spr '08 Prof

20 0. Use vectors to show that a parallelogram is a rectangle when its diagonals have equal length.



$$|\underline{u}+\underline{v}| = |\underline{u}-\underline{v}| \iff$$

$$(\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) = (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v}) \iff$$

$$\underline{u}\cdot\underline{u} + 2\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v} = \underline{u}\cdot\underline{u} - 2\underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v} \iff$$

$$\underline{u}\cdot\underline{v} = 0 \Rightarrow \underline{u} \perp \underline{v}$$

20 1. The plane curve C has polar equation $r = e^{\varepsilon\theta}$, where ε is a positive constant.

10 a) Find the equation of the tangent line at point $(x, y) = (1, 0)$.

$$x(\theta) = e^{\varepsilon\theta}\cos\theta, \quad y(\theta) = e^{\varepsilon\theta}\sin\theta$$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{\varepsilon e^{\varepsilon\theta}\sin\theta + e^{\varepsilon\theta}\cos\theta}{\varepsilon e^{\varepsilon\theta}\cos\theta - e^{\varepsilon\theta}\sin\theta}$$

$$(x, y) = (1, 0) \iff \theta = 0 \qquad \frac{dy}{dx}(1) = \frac{1}{\varepsilon}$$

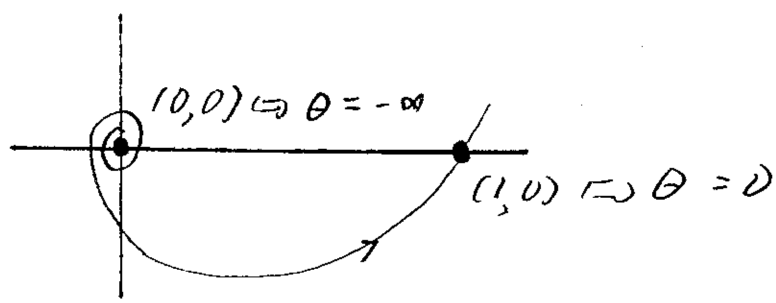
$$\boxed{y - 0 = \frac{1}{\varepsilon}(x - 1)}$$

10 b) Find the arclength between $(0,0)$ and $(1,0)$.

$$s = \int_{-\infty}^{0} \sqrt{r'^2 + r^2}\, d\theta =$$

$$\int_{-\infty}^{0} \sqrt{\varepsilon^2 e^{2\varepsilon\theta} + e^{2\varepsilon\theta}}\, d\theta =$$

$$\sqrt{1+\varepsilon^2} \int_{-\infty}^{0} e^{\varepsilon\theta}\, d\theta = \boxed{\frac{\sqrt{1+\varepsilon^2}}{\varepsilon} = s}$$



18 2. Find the point on the line through $(1,1,1)$ and $(1,0,-1)$ closest to the origin $(0,0,0)$.

$$\underline{r} = \langle 1,1,1\rangle + t\langle 0,-1,-2\rangle = \langle 1, 1-t, 1-2t\rangle$$

$$|\underline{r}|^2 = 3 - 6t + 5t^2$$

$$\frac{d}{dt}|\underline{r}|^2 = -6 + 10t \Rightarrow t = \frac{3}{5}$$

$$\underline{r}_c = \langle 1,1,1\rangle + \frac{3}{5}\langle 0,-1,-2\rangle$$

$$\boxed{\underline{r}_c = \left\langle 1, \frac{2}{5}, -\frac{1}{5}\right\rangle}$$

Find $\vec{r}$ 5 pts

$|\vec{r}|$ 3 pts

Find t_{min} pts

Get the pt: 2 pts

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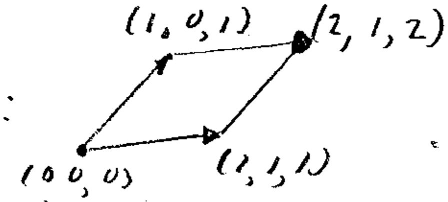
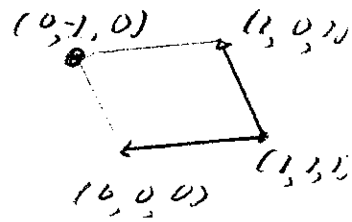
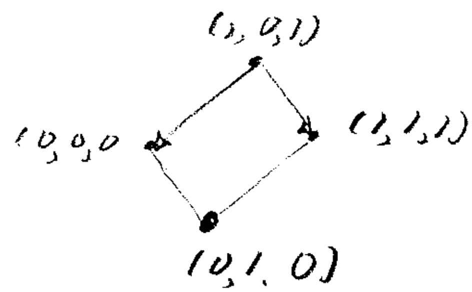
15 3. The points $(0,0,0)$, $(1,1,1)$, $(1,0,1)$ are three vertices of a parallelogram. What is the fourth vertex? What is the area of this parallelogram?

4th vertex:







Take $(2,1,2)$ case: $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \underline{i} - \underline{k}$

$A = |\underline{i} - \underline{k}| = \sqrt{2}$.

30 4. The trajectory of a particle has displacement from the origin, $\underline{r}(t) = \cos t\, \underline{i} + (\cos t + \sin t)\underline{j} + \cos t\, \underline{k}$.

10 a) At what t in $[0, 2\pi)$ do we have $\underline{r}'(t)$ orthogonal to $\underline{r}(t)$?

$$\underline{r}'(t) = -\sin t\, \underline{i} + (\cos t - \sin t)\underline{j} - \sin t\, \underline{k}$$

$$\underline{r} \cdot \underline{r}' = -2\sin t \cos t + \cos^2 t - \sin^2 t =$$

$$\cos 2t - \sin 2t = 0 \text{ if}$$

$$2t = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4} \iff$$

$$t = \frac{\pi}{8}, \frac{\pi}{8} + \frac{\pi}{2}, \frac{\pi}{8} + \pi, \frac{\pi}{8} + \frac{3\pi}{2}$$

b) What are the maximum and minimum distances of the particle from the origin?

$|\underline{r}|^2 = \cos^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t + \cos^2 t$

$= 1 + 2\cos^2 t + 2\cos t \sin t$

$= 2 + \cos 2t + \sin 2t.$

$t = \frac{\pi}{8} \Rightarrow |\underline{r}|^2 = 2 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2 + \sqrt{2}$ max

$t = \frac{5\pi}{8} \Rightarrow |\underline{r}|^2 = 2 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 2 - \sqrt{2}$ min

10 c) The trajectory lives in a plane containing the origin. What is the equation of this plane?

$$\underline{r} = \underbrace{(\underline{i}+\underline{j}+\underline{k})}\cos t + \sin t\,\underbrace{\underline{j}}$$

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\underline{i} + \underline{k}$$

$$\boxed{x - z = 0.}$$