(09) Final Exam Solutions

Note Title 12/16/2009

1.

(a) If $[A, H] = 0$ then

$$\langle a_\alpha | H | a_\beta \rangle = 0 \text{ if } a_\alpha \neq a_\beta$$

Where $H|a_\alpha\rangle = a_\alpha |a_\alpha\rangle$

If the eigenvalue of A is non-degenerate then H is diagonalized under the eigen basis of A

(b)

$$\vec{J}_1 + \vec{J}_2 = \begin{cases} J_1 + J_2 & 2(J_1+J_2)+1 \\ \vdots & \vdots \\ |J_1 - J_2| & 2|J_1-J_2|+1 \end{cases}$$

# of States

Example

$$\vec{\tfrac{1}{2}} + \vec{\tfrac{1}{2}} = \begin{cases} 1 & 3 \text{ states} \\ 0 & 1 \text{ states} \end{cases}$$

(c) $\langle \psi | H | \psi \rangle \geq E_0$ ← true ground state energy

(ψ: arbitrary normalized state)

Pf. $|\psi\rangle = \sum_\alpha C_\alpha |\phi_\alpha\rangle$

↑ true eigen state of $\mathcal{H}$

$$\langle \psi | H | \psi \rangle = \sum_\alpha E_\alpha |C_\alpha|^2$$

$$\geq \sum_\alpha E_0 |C_\alpha|^2 = E_0$$

2. (a) $|0\rangle \otimes |n+1\rangle \equiv |a\rangle$

$|e\rangle \otimes |n\rangle \equiv |b\rangle$

$$H = \begin{matrix} a \\ b \end{matrix} \begin{pmatrix} E_0 + (n+1)\hbar\omega & -i|g|\sqrt{n+1} \\ i|g|\sqrt{n+1} & E_0 + \Delta + n\hbar\omega \end{pmatrix}$$

(b)

$$\lambda = \left(E_0 + (n+1)\hbar\omega\right) \pm (n+1)|g|^2$$

eigenstates $|\psi_\pm\rangle \equiv \left(\pm i|0\rangle \otimes |n+1\rangle + |e\rangle \otimes |n\rangle\right) / \sqrt{2}$

(c) There is no change to part (a)

$$\lambda = E_0 + \frac{\Delta}{2} + (n+\tfrac{1}{2})\hbar\omega \pm \sqrt{\left(\frac{\Delta}{2} - \frac{1}{2}\hbar\omega\right)^2 + (n+1)|g|^2}$$

eigenstates $$\frac{i\left(\frac{1}{2}(\hbar\omega - \Delta) \pm \sqrt{\frac{1}{4}(\hbar\omega-\Delta)^2}\right)|0\rangle \otimes |n+1\rangle + |e\rangle \otimes |n\rangle}{\sqrt{1 + \left(\frac{1}{2}(\hbar\omega-\Delta) \pm \frac{1}{2}|\hbar\omega-\Delta|\right)^2}}$$

where $$\tan\theta = \frac{|g|(n+1)}{\frac{1}{2}(\hbar\omega-\Delta)}$$

(d) $t=0$ initial state $|e\rangle\otimes|0\rangle$

In this case we are dealing with $n=0$

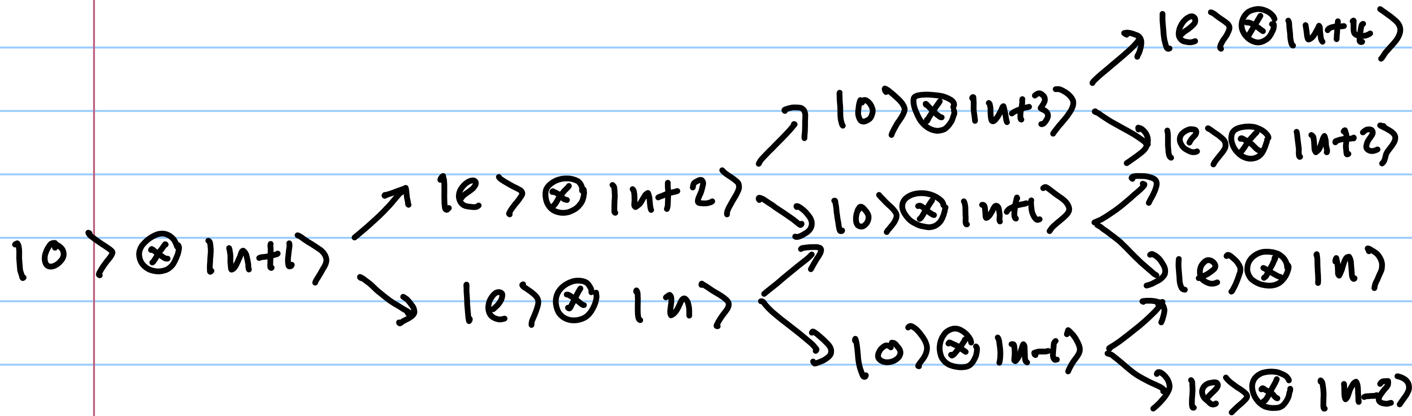
$$|\psi(0)\rangle = |e\rangle\otimes|0\rangle = \frac{1}{\sqrt{2}}\left(|\psi_+\rangle + |\psi_-\rangle\right)$$

where $$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(\pm i|0\rangle\otimes|1\rangle + |e\rangle\otimes|0\rangle\right)$$

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i|g|t/\hbar}|\psi_+\rangle + e^{i|g|t/\hbar}|\psi_-\rangle\right)$$

(e) We can no longer solve the problem

because



This form a never ending chain !

3.

(a) $\frac{dN_e}{dt} = -N_e K(n+1) + (N-N_e)Kn$

(b) $\because$ no photon initially, all atom are excited $\Rightarrow$ all photons are created by the decay ! $\therefore N_0 = n = N-N_e$

$\therefore \frac{dN_e}{dt} = -KN_e(N-N_e+1) + K(N-N_e)(N-N_e)$

(c) steady state $\Rightarrow \frac{dN_e}{dt} = 0$

$\therefore \quad N_e(N-N_e+1) = (N-N_e)^2$

$$N_e N - N_e^2 + N_e = N^2 + N_e^2 - 2NN_e$$

$$\Rightarrow \quad N_e = \frac{1}{4}\left(1+3N - \sqrt{1+6N+N^2}\right)$$

as $N \to \infty \quad N_e \to \frac{N}{2}$

(d) $\frac{dN_e}{dt} = -K\left(N_e\right)\left(N_o+1\right)(n+1) + K$

$$(N_e+1)\, N_o\, n$$

Again $n = N_o = N - N_e$

$$\frac{dN_e}{dt} = -KN_e(N-N_e+1)^2 + K(N_e+1)(N-N_e)^2$$

(e) $\frac{dN_e}{dt} = 0 \Rightarrow \quad N_e(N-N_e+1)^2 = (N_e+1)(N-N_e)^2$

$$N_e = \frac{1}{6}\left(1+4N - \sqrt{1+8N+4N^2}\right)$$

as $N \to \infty$

$$N_e \to \frac{1}{3} N$$

4.(a) initial State $|g\rangle \otimes |1\rangle$

final state $|k\rangle \otimes |0\rangle$

$$\langle f | V | i \rangle = \alpha$$

$$W = \frac{2\pi}{\hbar} \int d\Omega \int \frac{k^2 dk}{(2\pi)^3} |\alpha|^2 \delta\left(\frac{\hbar^2 k^2}{2m} - (-\Delta) - \hbar\omega\right)$$

$$= |\alpha|^2 \frac{2\pi}{\hbar} 4\pi \int \frac{k^2 dk}{(2\pi)^3} \delta\left(\frac{\hbar^2 k^2}{2m} + \Delta - \hbar\omega\right)$$

$$= \frac{2\pi}{\hbar} \frac{4\pi |\alpha|^2}{8\pi^3} \int dk\, k^2 \frac{1}{\left(\frac{\hbar^2 k}{m}\right)} \delta\left(k - \sqrt{\frac{2m}{\hbar^2}(\hbar\omega - \Delta)}\right)$$

$$= \frac{2\pi}{\hbar} \frac{4\pi |\alpha|^2}{8\pi^3} \frac{m}{\hbar^2} \sqrt{\frac{2m}{\hbar^2}(\hbar\omega - \Delta)}$$

(b) $\hbar\omega_{\text{threshold}} = \Delta$

5. $$(\vec{S}_n \cdot \vec{S}_N) = \left[(\vec{S}_n + \vec{S}_N)^2 - S_n^2 - S_N^2\right]/2$$

$$= \frac{1}{2}\left[(\vec{S}_n + \vec{S}_N)^2 - 3\frac{\hbar^2}{4} - 2\hbar^2\right]$$

$$\vec{S}_n + \vec{S}_N = \vec{\tfrac{1}{2}} + \vec{1} = \begin{cases} 3/2 \\ 1/2 \end{cases}$$

$$\vec{S}_n \cdot \vec{S}_N = \begin{cases} \frac{1}{2}\left(\frac{15}{4}\hbar^2 - \frac{11}{4}\hbar^2\right) = \frac{1}{2}\hbar^2 \\ \frac{1}{2}\left(\frac{3}{4}\hbar^2 - \frac{11}{4}\hbar^2\right) = -\hbar^2 \end{cases}$$

$$\therefore \left(\frac{d\sigma}{d\Omega}\right)_{S_{tot}=\frac{3}{2}} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|C_a + \tfrac{1}{2}C_b\right|^2 (\sqrt{\pi}d)^3 e^{-\frac{1}{4}\tilde{q}^2 d^2}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{S_{tot}=\frac{1}{2}} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|C_a - C_b\right|^2 (\sqrt{\pi}d)^3 e^{-\frac{1}{4}\tilde{q}^2 d^2}$$