Note Title 10/23/2009

1. a) The ground state has $j = \frac{1}{2} \left(= \vec{0} + \frac{\vec{1}}{2}\right)$.

It is 2-fold degenerate $2 = 2(\frac{1}{2}) + 1$

This 2-fold degeneracy is caused by the 2 possible value of the electron spin S_z.

b) In the presence of the hyperfine interaction we need to take the nuclear spin into consideration. The Hilbert space is now spanned by

$$\underbrace{|\pm\tfrac{1}{2}}_{\text{electron spin}}, \underbrace{\pm\tfrac{1}{2}\rangle}_{\text{nuclear spin}}$$

hence is 4-dimensional

With the hyperfine interaction

$$V = K \vec{S}_p \cdot \vec{S}_e$$
$$= \frac{K}{2}\left[\left(\vec{S}_p + \vec{S}_e \right)^2 - S_p^2 - S_e^2 \right]$$

Because both the nuclear spin and the electron spin are $\frac{1}{2}$

$$S_p^2 = \frac{3}{4}\hbar^2 = S_e^2$$

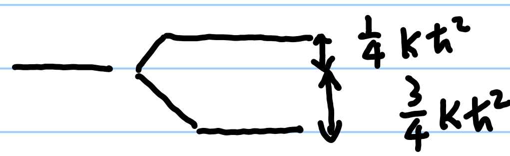
$\vec{S}_p + \vec{S}_e$ has two possible values (recall the rules of angular momenta addition)

$$\vec{\tfrac{1}{2}} + \vec{\tfrac{1}{2}} = \vec{1} \text{ or } \vec{0}$$

Thus $(\vec{S}_p + \vec{S}_e)^2$ has two possible values

$2\hbar^2$ or 0

So with the hyperfine interaction the original ground state energy level is split as shown



2. We use the variational ansatz

$$e^{-bx^2}$$

$$\langle E \rangle = \frac{\int_{-\infty}^{\infty} dx\, e^{-bx^2}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right) e^{-bx^2} - V_0 \int_{-1/2}^{1/2} dx\, e^{-2bx^2}}{\int_{-\infty}^{\infty} dx\, e^{-2bx^2}}$$

$$= \frac{\frac{\hbar^2}{2m}\sqrt{\frac{\pi}{2}}\sqrt{b} - V_0 L + O(bL^3)}{\sqrt{\frac{\pi}{2b}}}$$

$$= \frac{\hbar^2 b}{2m} - V_0 \sqrt{\frac{2}{\pi}} L \sqrt{b} + O(b^{3/2} L^3)$$

$$\frac{\partial E}{\partial b} = 0 \qquad \frac{\hbar^2}{2m} = V_0 \sqrt{\frac{2}{\pi}} \frac{1}{2\sqrt{b}} + O(b^{1/2} L^3)$$

as $V_0 \to 0$ the solution is $b^* = \dfrac{2L^2 m^2 V_0^2}{\hbar^4 \pi}$

$\langle E \rangle_{b^*} = - \dfrac{L^2 m V_0^2}{\hbar^2 \pi}$ therefore the ground state energy is less than zero (hence is a bound state) for any non-zero V_0.

3. For 2D square well the eigen energies are

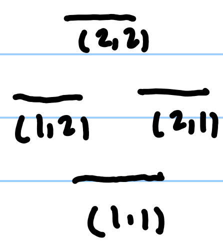
$$E(n_1, n_2) = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 (n_1^2 + n_2^2)$$

The eigenfunctions are

$$\psi_{(n_1,n_2)}(x,y) = \left(\frac{2}{L}\right) \cos\left(\frac{\pi}{L} x\right) \cos\left(\frac{\pi}{L} y\right)$$

Therefore in the absence of perturbation the lowest 4 energy levels are

$$E(1,1),\ E(1,2),\ E(2,1),\ E(2,2)$$



For the top and bottom levels we just need to calculate $\langle \psi_{11} | V | \psi_{11} \rangle$ & $\langle \psi_{22} | V | \psi_{22} \rangle$ to get the energy correction.

$$\langle \psi_{11} | V | \psi_{11} \rangle = V_0 \int_0^{L/2} dx \int_0^{L/2} dy \left(\frac{2}{L}\right)^2 \cos^2\left(\frac{\pi x}{L}\right) \cos^2\left(\frac{\pi y}{L}\right)$$

$$= V_0 \frac{4}{L^2} \left(\frac{L}{4}\right)^2 = \frac{V_0}{4}$$

$$\langle \psi_{22} | V | \psi_{22} \rangle = V_0 \int_0^{L/2} dx \int_0^{L/2} dy \left(\frac{2}{L}\right)^2 \cos^2\left(\frac{2\pi x}{L}\right) \cos^2\left(\frac{2\pi y}{L}\right)$$

$$= V_0 \frac{4}{L^2} \left(\frac{L}{4}\right)^2 = \frac{V_0}{4}$$

For the 2 mid levels we need to do degenerate perturbation calculation.

We set up the 2x2 matrix

$$V = \begin{pmatrix} \langle 12|V|12\rangle & \langle 12|V|21\rangle \\ \langle 21|V|12\rangle & \langle 21|V|21\rangle \end{pmatrix}$$

$$\simeq \begin{pmatrix} \frac{V_0}{4} & \frac{V_0}{3\pi} \\ \frac{V_0}{3\pi} & \frac{V_0}{4} \end{pmatrix}$$

The eigenvalues are $\frac{V_0}{4} \pm \frac{V_0}{3\pi}$

so to $O(V_0)$

$$E_1 \rightarrow \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2(1+1) + \frac{V_0}{4}$$

$$E_2 \rightarrow \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2(2+1) + \frac{V_0}{4} - \frac{V_0}{3\pi}$$

$$E_3 \rightarrow \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2(2+1) + \frac{V_0}{4} + \frac{V_0}{3\pi}$$

$$E_4 \to \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2(2+2) + \frac{V_0}{4}$$

4.

The idea of differential perturbation theory is to break a strong perturbation into many small pieces, i.e.,

$$H_0+V = \left\{\left[(H_0+\varepsilon V)+\varepsilon V\right]+\varepsilon V\right\}+\cdots$$

So that at each step we have an infinitesimal perturbation εV. We simply use 1st order non-deg perturbation theory to get the energy and wavefunctions. The formulae

$$\mathcal{H} = \mathcal{H}_0 + t\, V$$

is $$\dot{E}_n = \langle \phi_n(t) | V | \phi_n(t) \rangle$$

$$|\dot{\phi}_n(t)\rangle = \sum_{m\neq n} |\phi_m(t)\rangle \frac{\langle \phi_m(t)|V|\phi_n(t)\rangle}{\varepsilon_n(t)-\varepsilon_m(t)}$$