EECS 20N: Structure and Interpretation of Signals and Systems MIDTERM 1
Department of Electrical Engineering and Computer Sciences 18 February 2010
UNIVERSITY OF CALIFORNIA BERKELEY

LAST Name Scalia FIRST Name Tim E.

Lab Time In what reference frame?

- **(10 Points)** Print your name and lab time in legible, block lettering above AND on the last page where the grading table appears.
- This exam should take up to 70 minutes to complete. You will be given at least 70 minutes, up to a maximum of 80 minutes, to work on the exam.
- **This exam is closed book.** Collaboration is not permitted. You may not use or access, or cause to be used or accessed, any reference in print or electronic form at any time during the exam, except one double-sided $8.5" \times 11"$ sheet of handwritten notes having no appendage. Computing, communication, and other electronic devices (except dedicated timekeepers) must be turned off. Noncompliance with these or other instructions from the teaching staff—*including, for example, commencing work prematurely or continuing beyond the announced stop time*—is a serious violation of the Code of Student Conduct. Scratch paper will be provided to you; ask for more if you run out. You may not use your own scratch paper.
- **The exam printout consists of pages numbered 1 through 6.** When you are prompted by the teaching staff to begin work, verify that your copy of the exam is free of printing anomalies and contains all of the six numbered pages. If you find a defect in your copy, notify the staff immediately.
- Please write neatly and legibly, because *if we can't read it, we can't grade it.*
- For each problem, limit your work to the space provided specifically for that problem. *No other work will be considered in grading your exam. No exceptions.*
- Unless explicitly waived by the specific wording of a problem, you must explain your responses (and reasoning) succinctly, but clearly and convincingly.
- We hope you do a *fantastic* job on this exam.

MT1.1 (30 Points) You may tackle the two parts of this problem in either order.

(a) (14 Points) A continuous-time signal x is defined by

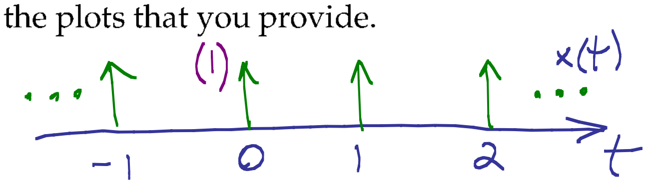
$$\forall t \in \mathbb{R}, \quad x(t) = \sum_{\ell=-\infty}^{+\infty} \delta(t-\ell).$$

Provide well-labeled plots of the signals $\widehat{x}$ and $\widetilde{x}$ defined by

$$\forall t \in \mathbb{R}, \quad \widehat{x}(t) = x(t/2) \quad \text{and} \quad \widetilde{x}(t) = x(2t).$$

You *must* explain how you arrive at the plots that you provide.

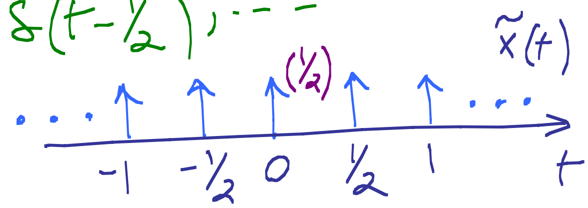
$$x(t) = \sum_{\ell=-\infty}^{\infty} \delta(t-\ell) = \cdots + \delta(t+1) + \delta(t) + \delta(t-1) + \cdots$$



$$\tilde{x}(t) = x(2t) = \sum_{\ell=-\infty}^{\infty} \delta(2t-\ell) = \sum_{\ell=-\infty}^{\infty} \delta\left(2\left(t-\ell/2\right)\right) = \cdots + \delta\left(2\left(t+\tfrac{1}{2}\right)\right) + \delta(2t) + \delta\left(2\left(t-\tfrac{1}{2}\right)\right) + \cdots$$

$\tilde{x}$ is a time-contracted version of x, by a factor of 2. However, caution is in order! The Dirac delta has the following time-scaling property: $\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$. Letting $\alpha = 2$, we have $\delta(2t) = \frac{1}{2}\delta(t)$, $\delta\left(2\left(t-\frac{1}{2}\right)\right) = \frac{1}{2}\delta\left(t-\frac{1}{2}\right), \ldots$

With this in mind, we can plot $\hat{x}$:



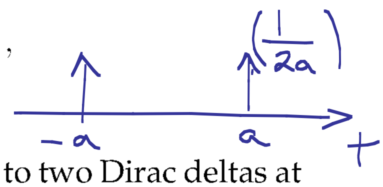
The signal $\hat{x}$ is a time-dilated version of x; the impulses in x separate from each other by a factor of 2. Again, we must use the time-scaling property of the Dirac delta, this time with $\alpha = 1/2$:

$$\hat{x}(t) = x(t/2) = \sum_{\ell=-\infty}^{\infty} \delta\left(\frac{t}{2}-\ell\right) = \sum_{\ell=-\infty}^{\infty} \delta\left(\frac{1}{2}(t-2\ell)\right) = \cdots + \delta\left(\frac{1}{2}(t+2)\right) + \delta\left(\frac{1}{2}t\right) + \delta\left(\frac{1}{2}(t-2)\right) + \cdots$$

Noting that $\delta(t/2) = 2\delta(t)$, $\delta\left(\frac{1}{2}(t+2)\right) = 2\delta(t+2)$, $\delta\left(\frac{1}{2}(t-2)\right) = 2\delta(t-2), \ldots$ the plot for $\hat{x}$ is as shown to the right

(b) (16 Points) Prove the identity

$$\delta(t^2 - a^2) = \frac{1}{2a}\left[\delta(t-a) + \delta(t+a)\right],$$



where $a > 0$.

First explain why the left-hand side (LHS) corresponds to two Dirac deltas at the locations specified by the right-hand side (RHS).

Next, integrate the LHS over a small neighborhood of a (e.g., from $a - \epsilon$ to $a + \epsilon$, for some infinitesimally small, but positive, ϵ) to establish the scaling factor $1/(2a)$ on the RHS. Explain why integrating the LHS around $-a$ yields the same scaling factor $1/(2a)$.

$\delta(t^2-a^2) = \delta\big((t-a)(t+a)\big)$ Note that an impulse is "active" (nonzero) anywhere its argument is zero. For $\delta(t^2-a^2)$, this occurs at $t = \pm a$, so we know $\delta(t^2-a^2)$ consists of two impulses, $\delta(t+a)$ and $\delta(t-a)$. Therefore,

$\delta(t^2-a^2) = A\delta(t-a) + B\delta(t+a)$ for some constants A and B, yet to be determined.

Let's find the strength of the impulse at $+a$. We integrate $\delta(t^2-a^2)$ from $a-\epsilon$ to $a+\epsilon$, for some infinitesimally-small, but positive ϵ.

$$\int_{a-\epsilon}^{a+\epsilon} \delta(t^2-a^2)\,dt = \int_{a-\epsilon}^{a+\epsilon} \delta\big[(t-a)\underbrace{(t+a)}\big]\,dt = \int_{a-\epsilon}^{a+\epsilon} \delta\big[(t-a)\underbrace{2a}\big]\,dt = \frac{1}{2a}\int_{a-\epsilon}^{a+\epsilon} \delta(t-a)\,dt = \frac{1}{2a} \Longrightarrow A = \frac{1}{2a}$$

In the neighborhood of $t=a$, replace "$t+a$" with $2a$

By time-scaling property

Integrate from $-a-\epsilon$ to $-a+\epsilon$ to find $B = 1/2a$ (also expected based on symmetry arguments)

MT1.2 (15 Points) Prove that $\left(e^{1/z}\right)^* = e^{1/z^*}$ for all values of the complex variable z where the exponentials are well-defined.

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \Longrightarrow e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{z^k},\ z \neq 0 \Longrightarrow \left(e^{1/z}\right)^* = \left(\sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{z^k}\right)^* \Longrightarrow$$

$$\left(e^{1/z}\right)^* = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{z^k}\right)^* = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{1}{(z^k)^*} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{z^*}\right)^k = e^{1/z^*}$$

MT1.3 (60 Points) You may tackle the two parts of this problem in either order.

(a) (20 Points) A continuous-time signal x is periodic with fundamental period p (note that $p > 0$). We sample this signal every T seconds to produce a discrete-time signal g as follows:

$$\forall n \in \mathbb{Z}, \quad g(n) = x(nT).$$

Is the signal g guaranteed to be periodic? If yes, explain your reasoning. If not, describe the conditions (if any exist) that guarantee g to be periodic.

For g to be periodic, we must have a positive integer N such that $g(n+N) = g(n)$, for all n. That is, we must have

$g(n+N) = x((n+N)T) = x(nT + NT) = g(n) = x(nT)$.

The equality holds if, and only if, $NT = Kp$, $K \in \{1, 2, 3, \ldots\}$

That is, $T = \frac{K}{N} p$.

We now have our answer:

g is periodic if, and only if, the sampling period T is a rational multiple of the period p of the CT signal x.

If T is not a rational multiple of p, then the DT signal g is not periodic.

Upshot: g is NOT guaranteed to be periodic.

(b) (40 Points) A periodic discrete-time signal h has fundamental period p (note that $p \in \{1, 2, 3, \ldots\}$). A related signal q is produced by sampling h every N samples. That is,

$$\forall n \in \mathbb{Z}, \quad q(n) = h(nN),$$

where $N \in \{2, 3, 4, \ldots\}$. Determine the period r of the signal q for each of the following cases:

$h(n+3) = h(n), \ \forall n \in \mathbb{Z}$

(i) $p = 3$ and $N = 5$. $\rightarrow q(n) = h(5n), \ \forall n \in \mathbb{Z}$

$q(n+r) = h(5(n+r)) = h(5n+5r) = h(5n) = q(n) \Rightarrow 5r = 3K, \ K \in \{1, 2, \ldots\}$
$\Rightarrow r = \frac{3}{5}K \Rightarrow$ Smallest acceptable K is $5 \Rightarrow r = \frac{3}{5} \cdot 5 = 3$
$\Rightarrow q$ is periodic with fundamental period $r = 3$.

$h(n+4) = h(n), \ \forall n$

(ii) $p = 4$ and $N = 6$. $\rightarrow q(n) = h(6n), \ \forall n$

$q(n+r) = h(6(n+r)) = h(6n+6r) = h(6n) = q(n) \Rightarrow 6r = 4K,$
$K \in \{1, 2, 3, \ldots\} \Rightarrow r = \frac{4}{6}K = \frac{2}{3}K \Rightarrow$ Smallest acceptable
K is 3, which makes $r = 2 \Rightarrow q$ is periodic with period 2

Is the signal q guaranteed to be periodic regardless of p and N, where $p \in \{1, 2, 3, \ldots\}$ and $N \in \{2, 3, 4, \ldots\}$? Explain your reasoning succinctly, but clearly and convincingly.

Yes, the signal q is guaranteed to be periodic. This is because N is always a rational multiple of p. In particular,
$q(n+r) = h(N(n+r)) = h(nN + Nr) = h(nN) = q(n) \Rightarrow Nr = Kp \Rightarrow$
$r = \frac{p}{N}K$. We're guaranteed that a positive integer K exists such that $r = \frac{p}{N}K$ is a positive integer. Clearly, $K = N$ does the job, so we can conclude that the period of p is at most $r = p$. The fundamental period r is exactly p, if N and p are relatively prime (as in part a(i)). Otherwise, $r < p$ (as in a(ii)).

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Lab Time In what reference frame?

You may or may not find the following information useful:

$$\delta(at) = \frac{1}{|a|}\delta(t).$$

$$e^z = \sum_{k=0}^{+\infty} \frac{z^k}{k!}.$$

Problem	Points	Your Score
Name	10	10
1	30	30
2	15	15
3	60	60
Total	**115**	115