Problem 1 solution

1 Part a

1.1 solution

The easiest way to calculate the potential is to take advantage of the effective spherical symmetry of infinitesimal charge elements

$$
\begin{aligned}
-\int_l \mathbf{E}\cdot \mathbf{dl} &= -\int_l \mathbf{dl}\cdot \int_Q \frac{kdq}{r^2}\hat{\mathbf{r}} \\
&= -\int_Q kdq \int_\infty^r \frac{\mathbf{dr}\cdot\hat{\mathbf{r}}}{r^2} = \int_Q \frac{kdq}{r}
\end{aligned}
$$

The apparent cylindrical symmetry of the problem encourages cylindrical coordinates, in which the charge element is $dq = \sigma\rho d\rho d\theta$, and $r = \sqrt{\rho^2 + a^2}$

$$
\begin{aligned}
k\int_0^a d\rho \int_0^{2\pi} \frac{\sigma\rho d\rho d\theta}{\sqrt{\rho^2+z^2}} &= k\int_{z^2}^{z^2+a^2} \frac{\pi\sigma du}{u^{\frac{1}{2}}} \\
= 2\pi\sigma k u^{\frac{1}{2}}|_{z^2}^{z^2+a^2} &= 2\pi k\sigma\left(\sqrt{z^2+a^2}-\sqrt{z^2}\right) \\
&= \frac{\sigma}{2\epsilon}k\left(\sqrt{z^2+a^2}-z\right)
\end{aligned}
$$

The u substitution being $u = z^2 + a^2 \implies du = zdz$

1.2 rubric

3 points for recognizing cylindrical symmetry.
3 points for distinguishing r from ρ(!!), writing r correctly, writing dq correctly, and setting up the integral
2 for calculating correctly and reporting the potential as a scalar

2 part b

$\mathbf{E} = -\nabla V$. Since there is only z dependence

$$
\begin{aligned}
\mathbf{E} &= -\frac{\partial V}{\partial z}\hat{\mathbf{z}} \\
&= \frac{\sigma}{2\epsilon}\left(1-\frac{z}{\sqrt{z^2+a^2}}\right)\hat{\mathbf{z}}
\end{aligned}
$$

2.1 rubric

4 points for setting up the calculation correctly
3 points for recognizing the potential only has dependence z. (Some thought there was a dependence, but a parametrizes the radius of the disk, and is not a coordinate).
1 point for including vector direction of $\mathbf{E}$

3 part c

By superposition,

$$\begin{aligned}
\mathbf{E}_{planewithaholeinit} &= \mathbf{E}_{plane} - \mathbf{E}_{disk} \\
&= \frac{\sigma}{2\epsilon}\hat{\mathbf{z}} - \frac{\sigma}{2\epsilon}\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right)\hat{\mathbf{z}} \\
&= \frac{\sigma}{2\epsilon}\frac{z}{\sqrt{z^2 + a^2}}\hat{\mathbf{z}}
\end{aligned}$$

3.1 rubric

5 points for seeing superposition
2 points for computing properly
1 point for remembering the $\hat{\mathbf{z}}$

1 Problem 2

(a)

[8pts for (a)] In terms of R and R', the resistance from A to B can be decomposed as follows:

$$\text{Series[} R',\ \text{Parallel[} R',\ \text{Series[} R,\ R' \text{]]]}.$$

Thus, the effective resistance from A to B is given by

$$R_{\text{tot}} = R' + \frac{1}{\frac{1}{R'} + \frac{1}{R+R'}}.$$

[Up to 6pts for this expression, partial credit for a partially correct answer] We can manipulate this expression to obtain

$$\begin{aligned}
R_{\text{tot}} &= R' + \left(\frac{R + R' + R'}{R'(R+R')}\right)^{-1}, \\
&= R' + \frac{R'(R+R')}{R+2R'}, \\
&= \frac{R'(R+2R'+R+R')}{R+2R'}, \\
&= \frac{2R+3R'}{R+2R'} \cdot R'.
\end{aligned}$$

Setting $R_{\text{tot}} = R$ and solving for R' yields:

$$\begin{aligned}
R &= \frac{2R+3R'}{R+2R'} \cdot R', \\
R(R+2R') &= (2R+3R')R', \\
R^2 + 2R'R &= 2RR' + 3(R')^2, \\
(R')^2 &= R^2/3.
\end{aligned}$$

Hence

$$\boxed{R' = \frac{1}{\sqrt{3}} R}.$$

[2 pts for solving for R' correctly]

(b)

[12 pts for all of (b). This part can be solved in many different ways: the grading scheme reflects one particularly straightforward approach, but it is possible to get full credit if you solve the problem in a different manner. One complication is that the manipulations with square roots of 3 can be quite hairy: I've thus correspondingly dropped very few points for mistaken algebra, emphasizing the points on the physical concepts: Voltages across parallel branches are equal, and they distribute over a sequence of elements in series; current divides at a junction; Ohm's law applies to all the resistors; etc.]

We first calculate the answers in terms of R' and V, then use the result from (a) to express these in terms of V and R only. As a check, we know that the total power dissipated should be

$$P_{\text{tot}} = \frac{V^2}{R_{\text{tot}}} = \frac{V^2}{R}.$$

Since $R_{\text{tot}} = R$, we know that the total current flowing from A to B must be given by [1pt]

$$I_{\text{tot}} = \frac{V}{R}.$$

Denote by I_i, V_i and P_i the current flowing through resistor i, the voltage drop across it, and the power dissipated through it. Since $I_4 = I_{\text{tot}}$, we have

$$V_4 = I_{\text{tot}} R' = V\frac{R'}{R},$$

We then have [2pt]

$$P_4 = V_4^2/R' = V^2\frac{R'}{R^2}.$$

The voltage across the rest of the circuit is

$$V_{\text{rest}} = V - V_4 = V\frac{R-R'}{R}.$$

Since the rest of the circuit consists of resistors 1 and 2 and resistor 3 in parallel, we have [2pt]

$$V_3 = V_{1+2} = V_{\text{rest}}.$$

Thus [1pt],

$$P_3 = V_3^2/R' = V^2\frac{(R-R')^2}{R^2R'}.$$

To examine resistors 1 and 2, we note that $I_1 = I_2$ and $V_1 + V_2 = V_{1+2}$, so

$$I_1(R_1 + R_3) = V_{\text{rest}} = V\frac{R-R'}{R},$$

so [2pt]

$$I_1 = I_2 = V\frac{R-R'}{R(R+R')}.$$

Hence [1pt each],

$$P_1 = I_1^2 R = V^2 \frac{(R-R')^2}{R(R+R')^2},$$

and

$$P_2 = I_2^2 R' = V^2 \frac{(R-R')^2 R'}{R^2(R+R')^2}.$$

To express these more compactly, let

$$\phi := \frac{R'}{R} = \frac{1}{\sqrt{3}}.$$

Then

$$\begin{aligned} P_1 &= \frac{V^2}{R}\frac{(1-\phi)^2}{(1+\phi)^2}, \\ P_2 &= \frac{V^2}{R}\frac{(1-\phi)^2\phi}{(1+\phi)^2}, \\ P_3 &= \frac{V^2}{R}\frac{(1-\phi)^2}{\phi}, \\ P_4 &= \frac{V^2}{R}\phi. \end{aligned}$$

Simplifying these, we get [2 pts for correct final answers in terms of V and R only, 1 pt if any incorrect, 0 pts if all incorrect]

$$\begin{aligned} P_1 &= \left(7 - \frac{12}{\sqrt{3}}\right) \cdot \frac{V^2}{R} \approx 0.072\, V^2/R, \\ P_2 &= \left(\frac{7}{\sqrt{3}} - 4\right) \cdot \frac{V^2}{R} \approx 0.041\, V^2/R, \\ P_3 &= \left(\frac{4}{\sqrt{3}} - 2\right) \cdot \frac{V^2}{R} \approx 0.309\, V^2/R, \\ P_4 &= \frac{1}{\sqrt{3}} \cdot \frac{V^2}{R} \approx 0.577\, V^2/R. \end{aligned}$$

We can see that, indeed, $P_1 + P_2 + P_3 + P_4 = V^2/R$.

Solution 3

When two capacitors are connected to each other, the charge will redistribute. Before the redistrubution, charge is

$$Q_1 = C_1 V_1 \tag{1}$$
$$Q_2 = C_2 V_2 \tag{2}$$

after the redistribution, suppose the charge will be Q_1' and Q_2', corresponding voltages are V_1' and V_2', we will have

$$V_1' = V_2' = V' \tag{3}$$

and because of the 'island' effect, we have

$$Q_1' + Q_2' = Q_1 + Q_2 \tag{4}$$

therefore

(a)

$$C_1 V' + C_2 V' = C_1 V_1 + C_2 V_2 \tag{5}$$

solve for V' we get

$$V_1' = V_2' = V' = \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2} \tag{6}$$

(b) The charges are easy to get

$$Q_1' = C_1 V' = C_1 \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2} \tag{7}$$
$$Q_2' = C_2 V' = C_2 \frac{C_1 V_1 + C_2 V_2}{C_1 + C_2} \tag{8}$$

(c) When we connect the two capacitors in opposite, we can replace V_2 with $-V_2$, and treat them the same way we did above. Therefore

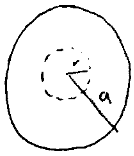
$$V'' = \frac{C_1 V_1 - C_2 V_2}{C_1 + C_2} \tag{9}$$

Positive sign of V'' means it is in the same direction as V_1, and vice versa.

(d) The charge on each capacitor is given by

$$Q_1'' = C_1 V'' = C_1 \frac{C_1 V_1 - C_2 V_2}{C_1 + C_2} \tag{10}$$
$$Q_2'' = C_2 V'' = C_2 \frac{C_1 V_1 - C_2 V_2}{C_1 + C_2} \tag{11}$$

(4) a) Assume a sphere of radius r has been built up, $r<a$



• The voltage going from ∞ to r given by

$$V = \frac{Q}{4\pi \varepsilon_0 r} \quad \text{assuming } V(\infty) = 0$$

(the result for voltage of spherical charge.)

where $Q = \rho \cdot \text{volume} = \rho \cdot \frac{4}{3}\pi r^3$

$$\Rightarrow V = \frac{\cancel{4\pi} r^{\cancel{3}\,2} \rho}{3 \cancel{4\pi} \varepsilon_0 \cancel{r}} = \frac{\rho}{3\varepsilon_0} r^2$$

$Q = \frac{4}{3}\pi r^3 \rho$

$dQ = 4\pi r^2 \rho \, dr$ (a layer of dQ)

• energy: $U = VQ$

$dU = V \, dQ$

$$dU = \frac{\rho}{3\varepsilon_0} r^2 (4\pi r^2 \rho) dr$$

$$U = \int_0^a \frac{4\pi\rho^2}{3\varepsilon_0} r^4 \, dr$$

$$= \frac{4\pi\rho^2}{3\varepsilon_0} \left. \frac{r^5}{5} \right|_0^a = \boxed{\frac{4\pi \rho^2 a^5}{15 \varepsilon_0}}$$

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b) $\rho = \dfrac{e}{\frac{4\pi}{3} b^3}$

$$U = \frac{4\pi \rho^2 b^5}{15 \varepsilon_0} = \frac{\cancel{4\pi} e^2 \cancel{b^5}}{\cancel{15}_{5} \left(\frac{\cancel{4\pi}}{3}\right)^2 b^{\cancel{6}} \varepsilon_0}$$

$$U = \frac{3q^2}{20\pi b \varepsilon_0}$$

Assume

$U = mc^2$

$$\Rightarrow mc^2 = \frac{3q^2}{20\pi b \varepsilon_0} \Rightarrow \boxed{b = \frac{3e^2}{20\pi \varepsilon_0 mc^2}}$$

$$\boxed{b = 1.68 \times 10^{-15} \text{ m}}$$

A couple comments

- You must somehow incorporate moving charges from ∞ to the ball. Taking $V = -\int_0^a E\, dr$ typically resulted in -4 points.

- Please remember $Q = \frac{4}{3}\pi r^3 \rho$ $\quad dQ = 4\pi r^2 \rho$
  One would ~~loose~~ lose 2 to 5 points based on severity of mistake.

- $\frac{1}{2}\frac{Q^2}{C}$, $\frac{1}{2}\int \rho V \, d(\text{volume})$, and $U = \int \frac{1}{2}\varepsilon_0 E^2 \, d(\text{volume})$
  all have a critical error in that they do not account for some of the energy. Use $U = QV$

- $U \neq V$ (potential energy is not equal to voltage)

- Lastly, some of you received -3 for having bad units. It is one thing to have a bad formula, it is quite another to assume a formula will magically give the correct units. ALWAYS CHECK YOUR UNITS, especially if you are unsure of your answer.