Operations Research II, IEOR161
University of California, Berkeley
Midterm Exam I, 2010

1. **[10+10+10]** There are 3 people on a sinking ship and they can jump one at a time onto a life-raft[1]. The ship will sink at a time determined by an exponential random variable with mean $1/\mu$ (i.e. rate μ). The time taken for the 1st person to escape is an exponential r.v. with rate λ_1, while the 2nd person takes an exponential r.v. with rate λ_2 and the 3rd is exponential with rate λ_3.

   (a) What is the probability that no passenger escapes?

   (b) What is the probability that all passengers escape?

   (c) If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ what is the expected number of people that escape?

Solution:
T: time taken for the ship to sink $\sim exp(\mu)$
T_i: time taken for the i^{th} person to escape $\sim exp(\lambda_i)$
N: number of people that escapes

a)

$$P(T < T_1) = \frac{\mu}{\mu + \lambda_1}.$$

b)

$$\begin{aligned}
P(T > T_1 + T_2 + T_3) &= P(T > T_1)P(T > T_1 + T_2 | T > T_1)P(T > T_1 + T_2 + T_3 | T > T_1 + T_2) \\
&= P(T > T_1)P(T > T_2)P(T > T_3) \\
&\quad \text{by memoryless of exponential distribution} \\
&= \frac{\lambda_1}{\mu + \lambda_1}\frac{\lambda_2}{\mu + \lambda_2}\frac{\lambda_3}{\mu + \lambda_3}.
\end{aligned}$$

c)

$$\mathbb{E}[N] = \sum_{i=0}^{3} iP(N = i),$$

where,

$$P(N = 1) = P(T > T_1, T < T_1 + T_2)$$

[1]That is, the 2nd and 3rd people wait in line until the first person jumps. If the ship is still floating, then the 2nd person attempts to leave, etc.

$$
\begin{aligned}
&= P(T > T_1)P(T < T_1 + T_2 | T > T_1) \\
&= P(T > T_1)P(T < T_2) \\
&\quad \text{by memoryless of exponential distribution} \\
&= \frac{\lambda}{\mu+\lambda}\frac{\mu}{\mu+\lambda}.
\end{aligned}
$$

similarly we can get,

$$
\begin{aligned}
P(N=2) &= \frac{\lambda}{\mu+\lambda}\frac{\lambda}{\mu+\lambda}\frac{\mu}{\mu+\lambda}, \\
P(N=3) &= \frac{\lambda}{\mu+\lambda}\frac{\lambda}{\mu+\lambda}\frac{\lambda}{\mu+\lambda},
\end{aligned}
$$

hence,

$$
\mathbb{E}[N] = \sum_{i=0}^{3} iP(N=i) = \frac{\lambda\mu}{(\mu+\lambda)^2} + 2\frac{\lambda^2\mu}{(\mu+\lambda)^3} + 3\frac{\lambda^3}{(\mu+\lambda)^3}
$$

2. **[15+15]** A train has just departed and the station is empty. New customers arrive according to a Poisson process with rate λ, and the next train arrives after an exponential time with rate μ. Assuming that the train has infinite capacity and instantaneously picks up waiting customers and leaves, and that the customer arrival process and the time until the next train arrival are independent:

   (a) What is the expected number of customers on the next train?

   (b) What is the variance of the number of customers on the next train?

   Solution:
   T: inter-arrival time of the train $\sim exp(\mu)$
   $N(t)$: arrival process of new customer $\sim Poi(\lambda)$

   a)

$$
\mathbb{E}[N(T)] = \mathbb{E}[\mathbb{E}[N(T)|T]] = \mathbb{E}[\lambda T] = \lambda\mathbb{E}[T] = \frac{\lambda}{\mu}.
$$

   b)

$$
\begin{aligned}
\mathbb{V}(N(T)) &= \mathbb{E}[\mathbb{V}(N(T)|T)] + \mathbb{V}(\mathbb{E}[N(T)|T]) \\
&= \mathbb{E}[\lambda T] + \mathbb{V}(\lambda T) \\
&= \lambda\mathbb{E}[T)] + \lambda^2\mathbb{V}(T) \\
&= \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu^2}
\end{aligned}
$$

3. **[10+10+20]** A service center has two servers. Server 1 takes a exponential time with rate μ_1 to serve customers while server 2 takes an exponential time with rate μ_2. Suppose that both servers are busy when you arrive, and that you will be served by the first available server (there is no one else waiting for service)

   (a) What is the expected time until you leave the system?

   (b) What is the expected time for the system to be cleared (assuming no arrivals occur after you)?

   (c) Assuming that future customers arrive according to a Poisson process with rate λ what is the probability that the 3 original customers are cleared before the next arrival?

Solution:

a)
T: time until you leave the system
W: your time waiting for a server
S: your time in service

$$
\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[W] + \mathbb{E}[S] \\
&= \frac{1}{\mu_1+\mu_2} + \mathbb{E}[\mathbb{E}[S|\text{which server becomes available first}]] \\
&= \frac{1}{\mu_1+\mu_2} + \left(\frac{1}{\mu_1}\frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_2}\frac{\mu_2}{\mu_1+\mu_2}\right) \\
&= \frac{3}{\mu_1+\mu_2},
\end{aligned}
$$

where W equals the minimum of the two servers' completion time.

b) This problem is different from the above as you may not be the last one to leave the system.
T: time taken for the system to clear (assuming no new arrival)
T_i: time taken for the i^{th} departure

$$
\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T_1] + \mathbb{E}[T_2] + \mathbb{E}[T_3] \\
&= \frac{1}{\mu_1+\mu_2} + \frac{1}{\mu_1+\mu_2} + \mathbb{E}[[\mathbb{E}[T_3|\text{which server finishes last}]] \\
&= \frac{2}{\mu_1+\mu_2} + \left(\frac{1}{\mu_2}\frac{\mu_1}{\mu_1+\mu_2} + \frac{1}{\mu_1}\frac{\mu_2}{\mu_1+\mu_2}\right)
\end{aligned}
$$

c)

T_λ: time taken to see the next arrival

$$
\begin{aligned}
P[T_\lambda > T_1 + T_2 + T_3] &= P(T_\lambda > T_1)P(T_\lambda > T_1 + T_2 | T_\lambda > T_1)P(T_\lambda > T_1 + T_2 + T_3 | T_\lambda > T_1 + T_2) \\
&= P(T_\lambda > T_1)P(T_\lambda > T_2)P(T_\lambda > T_3) \\
&\quad \text{by memoryless of exponential distribution} \\
&= \frac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2} \frac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2} \mathbb{E}[P(T_\lambda > T_3 | \text{which server finishes last})] \\
&= (\frac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2})^2 [\frac{\mu_1}{\lambda + \mu_1} \frac{\mu_2}{\mu_1 + \mu_2} + \frac{\mu_2}{\lambda + \mu_2} \frac{\mu_1}{\mu_1 + \mu_2}].
\end{aligned}
$$