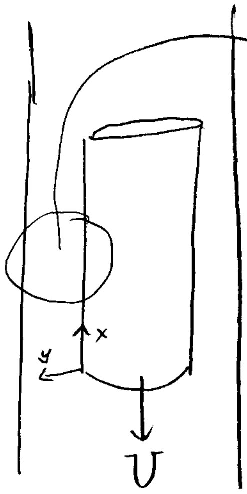


Assume:

Newtonian, incompressible

steady-state

Since $L >> R_2 - R_1$, assume flow in gap is fully-developed.

For ease of calculation, I will fix my local coord. syst. on falling cylinder.

$\Rightarrow$ Assume $v_x = v_x(y)$ only

$v_y = 0$, $v_z = 0$

Continuity

$$\cancel{\frac{\partial v_x}{\partial x}} + \cancel{\frac{\partial v_y}{\partial y}} + \cancel{\frac{\partial v_z}{\partial z}} = 0$$

$0 = 0$ ✓ satisfied

x-comp of N-S.

$$\rho\left(\cancel{\frac{\partial v_x}{\partial t}} + v_x \cancel{\frac{\partial v_x}{\partial x}} + \cancel{v_y}\frac{\partial v_x}{\partial y} + \cancel{v_z}\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \eta\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \cancel{\frac{\partial^2 v_x}{\partial z^2}}\right)$$

$$0 = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 v_x}{\partial y^2}\right)$$

$$0 = -\frac{\partial p}{\partial x} + \eta\frac{d^2 v_x}{dy^2}$$

ordinary deriv., since $v_x = v_x(y)$ only

y-comp of N-S

$$0 = -\frac{\partial p}{\partial y} \quad \Rightarrow \quad p \neq p(y)$$

z-comp of N-S

$$0 = -\frac{\partial p}{\partial z} \quad \Rightarrow \quad p \neq p(z)$$

$\Rightarrow \quad P = P(x)$ only

so x-comp. becomes

$$0 = -\frac{dP}{dx} + \eta \frac{d^2 v_x}{dy^2}$$

$$\underbrace{\frac{dP}{dx}}_{\text{function of } x \text{ only}} = \underbrace{\eta \frac{d^2 v_x}{dy^2}}_{\text{function of } y \text{ only}} = \text{constant} = \frac{\Delta P}{L}$$

$$\frac{d^2 v_x}{dy^2} = \frac{1}{\eta}\frac{\Delta P}{L}$$

$$\frac{dv_x}{dy} = \frac{1}{\eta}\frac{\Delta P}{L} y + C_1$$

$$v_x = \frac{1}{2\eta}\frac{\Delta P}{L} y^2 + C_1 y + C_2$$

In our coord. system, BC's are:

$$v_x(y=0) = 0$$

$$v_x(y=H) = U$$

substituting in

$$0 = 0 + 0 + C_2 \Rightarrow C_2 = 0$$

$$U = \frac{1}{2\eta}\frac{\Delta P}{L} H^2 + C_1 H$$

$$C_1 = \frac{U}{H} - \frac{1}{2\eta}\frac{\Delta P}{L} H$$

$$v_x(y) = \frac{1}{2\eta}\frac{\Delta P}{L} y^2 + \left(\frac{U}{H} - \frac{1}{2\eta}\frac{\Delta P}{L} H\right) y$$

$$v_x(y) = U\left(\frac{y}{H}\right) - \frac{H^2}{2\eta}\left(\frac{\Delta P}{L}\right)\left[\left(\frac{y}{H}\right) - \left(\frac{y}{H}\right)^2\right]$$

Use conservation of mass, as in the gear pump problem, to solve for $\frac{\Delta P}{L}$.

Here, the flow rate through the gap H must equal the rate of volume displacement by the cylinder. That is, the fluid the solid cylinder displaces as it falls must flow up thru the gap.

$$Q = \pi R_1^2 U = 2\pi R_1 \int_0^H v_x \, dy$$

$$= 2\pi R_1 \int_0^H \left[U\left(\frac{y}{H}\right) - \frac{H^2}{2\eta}\left(\frac{\Delta P}{L}\right)\left[\frac{y}{H} - \frac{y^2}{H^2}\right]\right] dy$$

$$= 2\pi R_1 \left[\frac{U}{H}\frac{1}{2}y^2 \Big|_0^H - \frac{H}{2\eta}\left(\frac{\Delta P}{L}\right)\frac{1}{2}y^2 \Big|_0^H + \frac{1}{2\eta}\left(\frac{\Delta P}{L}\right)\frac{1}{3}y^3 \Big|_0^H \right]$$

$$\pi R_1^2 U = 2\pi R_1 \left[\frac{1}{2}UH - \frac{1}{4}\frac{1}{\eta}\left(\frac{\Delta P}{L}\right)H^3 + \frac{1}{6}\frac{1}{\eta}\left(\frac{\Delta P}{L}\right)H^3 \right]$$

$$\cancel{\pi} R_1^{\cancel{2}} U = 2\cancel{\pi}\cancel{R_1} \left[\frac{UH}{2} - \frac{H^3}{12\eta}\left(\frac{\Delta P}{L}\right) \right]$$

$$UR_1 - UH = -\frac{H^3}{6\eta}\left(\frac{\Delta P}{L}\right)$$

$$UR_1\left[1 - \frac{H}{R_1}\right] = -\frac{H^3}{6\eta}\left(\frac{\Delta P}{L}\right)$$

$$\frac{\Delta P}{L} = -\frac{6\eta R_1 U}{H^3}\left[1 - \frac{H}{R_1}\right] \simeq -\frac{6\eta R_1 U}{H^3}$$

may be neglected since $\frac{H}{R_1} \ll 1$

$$\Rightarrow \boxed{v_x(y) = U\left(\frac{y}{H}\right) - \frac{H^2}{2\eta}\left(-\frac{6\eta R_1 U}{H^3}\right)\left[1 - \frac{H}{R_1}\right]\left(\frac{y}{H} - \frac{y^2}{H^2}\right)}$$

in moving coord. system.

To get velocity in <u>fixed</u> coord system, just add $-U$ to velocity field, i.e.

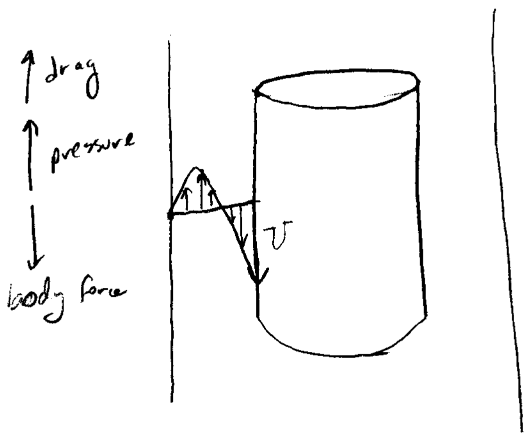
$$v_x(y) = -U\left[1-\frac{y}{H}\right] - \frac{H^2}{2\eta}\left(\frac{-6\eta R_1 U}{H^3}\right)\left[1-\frac{H}{R_1}\right]\left(\frac{y}{H}-\frac{y^2}{H^2}\right)$$

(now, at $y=0$, $v_x = -U$

at $y=H$, $v_x = 0$)

Now we write a force balance on the cylinder to relate η to U.

We are interested in the x-comp. of the force



The forces acting on cylinder are a body force, a drag force due to τ_{yx}, and a pressure force acting on ends of cylinder due to ΔP (note this includes gravitational force).

$$\sum F_x = 0 = -\underbrace{\pi R_1^2 L}_{\text{volume of fluid displaced}}\underbrace{(\rho_c - \rho)g}_{\text{body force includes buoyancy}} + \pi R_1^2\left[P(0) - P(L)\right]$$

difference in pressures acting on two ends of cylinders

$$+ \underbrace{2\pi R_1 L\, \tau_{yx}\Big|_{y=0}}_{\text{drag force}}$$

From part (a), we know $\dfrac{dP}{dx} = \dfrac{\Delta P}{L} = \dfrac{-6\eta R_1 U}{H^3}\left[1-\dfrac{H}{R_1}\right]$

$$\Rightarrow \quad P(x) = \frac{-6\eta R_1 U}{H^3}\left[1 - \frac{H}{R_1}\right] x + C_0$$

at $x = 0$, $P = P_0$ $\Rightarrow$ $C_0 = P_0 = P(0)$

$$\Rightarrow \quad P(L) = -\frac{6\eta R_1 U}{H^3}\left[1 - \frac{H}{R_1}\right] L + P(0)$$

$$\Rightarrow \quad P(0) - P(L) = \frac{6\eta R_1 U}{H^3}\left[1 - \frac{H}{R_1}\right] L \simeq \frac{6\eta R_1 U L}{H^3}$$

We can evaluate $\tau_{yx}\Big|_{y=0}$ from the velocity field:

$$\tau_{yx}\Big|_{y=0} = \eta \frac{dv_x}{dy}\Big|_{y=0} = \eta\left[\frac{U}{H} - \frac{H^2}{2\eta}\left(\frac{-6\eta R_1 U}{H^3}\right)\left(1 - \frac{H}{R_1}\right)\left(\frac{1}{H} - \frac{2y}{H^2}\right)\right]\Bigg|_{y=0}$$

$$= \eta\left[\frac{U}{H} + \frac{3R_1 U}{H}\left(1 - \frac{H}{R_1}\right)\left(\frac{1}{H}\right)\right]$$

$$= \frac{\eta U}{H}\left[1 + \frac{3R_1}{H}\left(1 - \frac{H}{R_1}\right)\right]$$

$$\simeq \frac{\eta U}{H}\left[1 + 3\frac{R_1}{H}\right] \simeq \frac{\eta U}{H}\left(\frac{3R_1}{H}\right) \simeq \frac{3\eta U R_1}{H^2}$$

The drag force is

$$F_{drag} = \tau_{yx}\Big|_{y=0} A = 2\pi R_1 L\left(\frac{3\eta U R_1}{H^2}\right) \leftarrow \text{answer to (b)}$$

Substituting into our force balance:

$$0 = -\pi R_1^2 L(\rho_c - \rho)g + \pi R_1^2\left(\frac{6\eta R_1 U L}{H^3}\right) + 2\pi R_1 L\left(\frac{3\eta U R_1}{H^2}\right)$$

Simplifying

$$0 = -(\rho_c - \rho)g + \frac{6\eta R_1 U}{H^3} + \frac{6\eta U}{H^2}$$

$$(\rho_c - \rho)g = \frac{6\eta R_1 U}{H^3}\left(1 + \frac{H}{R_1}\right) \simeq \frac{6\eta R_1 U}{H^3}$$

$$\Rightarrow \quad \boxed{\eta = \frac{(\rho_c - \rho)g H^3}{6 R_1 U}}$$

2. a). Assume Newtonian, incompressible fluid, steady, $Re \ll 1$ (creeping flow)

Because of symmetry, assume $\underline{v}$ and P are not functions of ϕ

Assume, $v_r = 0$, $v_\theta = 0$ & $v_\phi = v_\phi(r, \theta)$

$P = P(r, \theta)$ by symmetry

Continuity:

$$0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cancelto{0}{v_r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\cancelto{0}{v_\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\cancelto{0}{\frac{\partial}{\partial\phi}}\left(v_\phi\right)$$

$$0 = 0$$

b) r-comp of N-S

$$0 = -\frac{\partial P}{\partial r} \Rightarrow P \neq P(r)$$

θ-comp of N-S

$$0 = -\frac{1}{r}\frac{\partial P}{\partial\theta} \Rightarrow P \neq P(\theta)$$

$\Rightarrow$ $\underline{P = \text{constant}}$!

ϕ-comp of N-S

$$0 = \eta\left[\frac{1}{\cancel{r^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{\cancel{r^2}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\phi}{\partial\theta}\right) + \frac{1}{\cancel{r^2}\sin^2\theta}\cancelto{0}{\frac{\partial^2 v_\phi}{\partial\phi^2}} - \frac{v_\phi}{\cancel{r^2}\sin^2\theta}\right]$$

$$\boxed{0 = \frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\phi}{\partial\theta}\right) - \frac{v_\phi}{\sin^2\theta}}$$

$\underline{\text{BC's}}$: $v_\phi = 0$ at $r = kR$ (no slip)

$v_\phi = \Omega R\sin\theta$ at $r = R$

c) As in other creeping flow problems we solved, let's guess that the solution retains the same dependence on θ throughout the domain.

That is, let's try a solution of the form

$$v_\phi = f(r)\sin\theta$$

that satisfies BC's $\quad f(r=kR) = 0$

$\qquad\qquad\qquad\qquad f(r=R) = \Omega R$

substitute this into our circled equation above:

$$0 = \frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(f\sin\theta)\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}(f\sin\theta)\right) - \frac{f\sin\theta}{\sin^2\theta}$$

$$0 = \sin\theta\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \frac{f}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\cos\theta\right) - \frac{f}{\sin\theta}$$

$$0 = \sin\theta\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \frac{f}{\sin\theta}\left(-\sin^2\theta + \underbrace{\cos^2\theta}_{1-\sin^2\theta}\right) - \frac{f}{\sin\theta}$$

$$0 = \sin\theta\frac{d}{dr}\left(r^2\frac{df}{dr}\right) + \frac{f}{\sin\theta}\left(\cancel{1} - 2\sin^2\theta\right) - \cancel{\frac{f}{\sin\theta}}$$

$$\boxed{0 = \frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 2f}$$

BCs $\quad f(kR) = 0$

$\qquad f(R) = \Omega R$

Expanding:

$$r^2\frac{d^2f}{dr^2} + 2r\frac{df}{dr} - 2f = 0$$

Using hint provided: $\quad a = 1 \quad b = 2 \quad c = -2$

$$n(n-1) + 2n - 2 = 0$$

$$n^2 - n + 2n - 2 = 0$$

$$n^2 + n - 2 = 0$$

$$(n+2)(n-1) = 0 \quad\Rightarrow\quad n = -2,\ n = 1$$

Solution is $\quad f = C_1 r^{-2} + C_2 r^1$

Applying BC at $r = kR$:

$$0 = C_1 \frac{1}{(kR)^2} + C_2 (kR)$$

$$C_1 = -C_2 (kR)^3$$

Applying BC at $r = R$

$$\Omega R = -C_2 (kR)^3 \frac{1}{R^2} + C_2 R = -C_2 (k^3 R - R)$$

$$\Omega R = C_2 R (1 - k^3)$$

$$\Rightarrow \quad C_2 = \frac{\Omega}{1-k^3}$$

$$\Rightarrow \quad C_1 = -\frac{\Omega}{1-k^3} (kR)^3$$

$$f = -\frac{\Omega}{1-k^3} (kR)^3 \frac{1}{r^2} + \frac{\Omega}{1-k^3} r$$

$$f = \frac{\Omega}{1-k^3} \left(r - \frac{(kR)^3}{r^2} \right)$$

$$\Rightarrow \quad \boxed{V_\phi = \frac{\Omega}{1-k^3} \left(r - \frac{(kR)^3}{r^2} \right) \sin\theta}$$

d). largest inertial term discarded from r-comp is :

$$\frac{\rho V_\phi^2}{r} \sim \frac{\rho}{R} \left(\frac{\Omega R}{1-k^3} \right)^2 \sim \frac{\rho \Omega^2 R}{(1-k^3)^2}$$

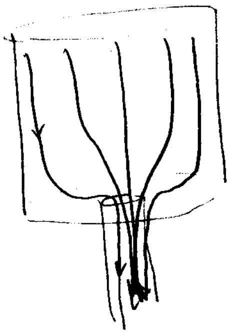
Viscous term is

$$\eta \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) \sim \eta \frac{1}{R^2} \frac{1}{R} \left(R^2 \frac{1}{R} \frac{\Omega}{1-k^3} R \right)$$

Viscous $\sim \eta \frac{1}{R} \frac{\Omega}{1-k^3}$

Ratio of $\frac{\text{inertial}}{\text{viscous}} \sim \frac{\rho \Omega^2 R}{(1-k^3)^2} \frac{R(1-k^3)}{\eta \Omega} \sim \frac{\rho \Omega R^2}{\eta (1-k^3)}$

We are interested in flow near the contraction plane, where streamlines will likely look like:



so we'll assume:

Newtonian
incompressible
steady
H is comparable to R_1

Axisymmetric $\Rightarrow$ no θ dependence

Assume

a)
$$\boxed{\begin{aligned} v_\theta &= 0 \\ v_r &= v_r(r,z) \\ v_z &= v_z(r,z) \end{aligned}}$$

$$P = P(r,z)$$

b) $v_r = 0$ at $r = R_1$ for $z < 0$
at $r = R_2$ for $z > 0$

$v_r = 0$ at $r = 0$ for all z

$v_z = 0$ at $r = R_1$ for $z < 0$
at $r = R_2$ for $z > 0$

v_z = finite at $r = 0$ OR $\frac{\partial v_z}{\partial r} = 0$ at $r = 0$

Note: $\left.\frac{\partial v_r}{\partial r}\right|_{r=0} \neq 0$ since $\left.\frac{\partial v_z}{\partial z}\right|_{r=0} \neq 0$ (see continuity).

c) r-comp: becomes:

$$\rho\left(v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z}\right) = -\frac{\partial P}{\partial r} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right]$$

θ-comp becomes:

$$0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} \Rightarrow P \neq P(\theta)$$

z - comp becomes

$$\rho\left(v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial P}{\partial z} + \eta\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) + \frac{\partial^2 v_z}{\partial z^2} \right]$$

d). From our boundary layer approx., we got

$$\frac{L_e}{D} \simeq 0.01\, Re$$

$$z \simeq 0.01\left(\frac{\rho \langle v \rangle D^2}{\eta} \right) = 0.01\left(\frac{4\rho\langle v \rangle R_2^2}{\eta} \right)$$

Basing your answer on Eqn 3.14a is ok too. . .