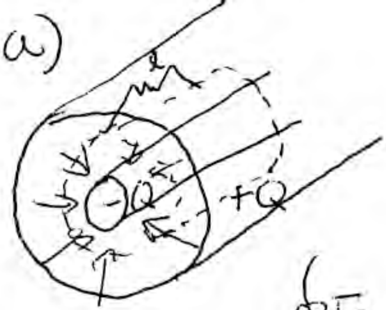
1. a)



Using Gauss's law, with a cylinder of length l:

$$\oint E \cdot da = |E| \oint da = |E| \cdot 2\pi r \cancel{l} = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{(-Q)}{L}\cancel{l}$$

$$|E| = -\frac{1}{\epsilon_0}\frac{Q}{L}\frac{1}{2\pi r} \quad (+3 \text{ pts})$$

By symmetry, we know it points axially inward:

$$\vec{E} = \frac{-Q}{2\pi r \epsilon_0 L}\hat{r} \quad (+2 \text{ pts})$$

Note: -1 pt. if you put $+Q$ on the inner cylinder.

b) Choose a reference pt. ~~other~~ at the inner radius a:

$$\underset{a<r<b}{V(r=a) - V(r)} = -\int_r^a \vec{E} \cdot d\vec{l} = \int_a^r E \cdot dl$$

Set $V(r=a) = 0$

$$V(r) = -\int_a^r \frac{-Q}{2\pi r' \epsilon_0 L}\hat{r}\cdot\hat{r}\, dr' = \int_a^r \frac{Q}{2\pi\epsilon_0 L}\frac{1}{r'}dr'$$

$$= \frac{Q}{2\pi\epsilon_0 L}\ln r'\Big|_a^r = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{r}{a}\right)$$

You could also choose your reference pt. at outer radius b:

$$V(r=b) - V(r) = -\int_r^b \vec{E}\cdot d\vec{l} \Rightarrow V(r) = \int_r^b \frac{-Q}{2\pi r' \epsilon_0 L}dr' = \frac{-Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{r}\right)$$

(+5 pts for either answer)

Using first ref pt:

c) $\Delta V = V(b) - V(a) = \frac{Q}{2\pi\epsilon_0 L}\left(\ln\left(\frac{b}{a}\right) - \cancel{\ln(1)}^{\,0}\right) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$

Using second ref. pt:

$$\Delta V = V(b) - V(a) = -\frac{Q}{2\pi\epsilon_0 L}\left(\ln(1) - \ln\left(\frac{b}{a}\right)\right) = \frac{Q}{2\pi\epsilon_0 L}\left(\ln\frac{b}{a}\right)$$

Note: These should have positive ΔV. And ΔV should be the same regardless of ref. pt.

$Q = CV$ $\quad Q = \underbrace{\frac{2\pi\epsilon_0 L}{\ln(b/a)}}_{C}\Delta V \Rightarrow C = \frac{2\pi\epsilon_0 L}{\ln(b/a)}$ (+5 pts)

d) $U = \frac{1}{2}\frac{Q^2}{C} = \frac{\frac{1}{2}Q^2\ln(b/a)}{2\pi\epsilon_0 L} = \frac{Q^2}{4\pi\epsilon_0 L}\ln(b/a)$ (+5 pts)

e) $U = \frac{1}{2}\epsilon_0\int E^2 dV = \frac{1}{2}\epsilon_0\int\frac{Q^2}{4\pi^2\epsilon_0^2 L^2}\left(\frac{1}{r'^2}\right)dV$

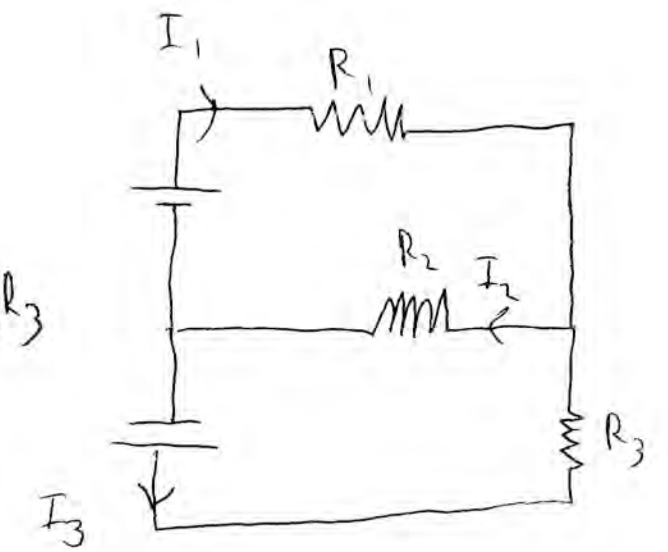
$= \frac{Q^2}{8\pi^2 L^2\epsilon_0}\int_0^L\int_a^b\int_0^{2\pi}\frac{1}{r'}\,d\theta\,dr'\,dl = \frac{Q^2}{8\pi^2 L^2\epsilon_0}\times L\times\int_a^b\int_0^{2\pi}\frac{1}{r'}\,dr'\,d\theta$

$= \frac{Q^2}{8\pi^2 L^2\epsilon_0}\times L\times 2\pi\times\int_a^b\frac{1}{r'}\,dr' = \frac{Q^2}{4\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$

It agrees as expected. (+5 pts)

3

$R_2 \ll R_1, R_3$



$\boxed{5 \text{ pts}}$

diagram & direction & justification

Kirchhoff's voltage rule

$\boxed{10 \text{ pts}}$

$$\mathcal{E}_1 - I_1 R_1 - I_2 R_2 = 0 \qquad (1)$$

$$\mathcal{E}_2 - I_3 R_3 - I_2 R_2 = 0 \qquad (2)$$

Kirchhoff's Junction rule

$$I_2 = I_3 + I_1 \qquad (3)$$

Put (3) into (2)

$$\mathcal{E}_2 - I_3 R_3 - (I_3 + I_1) R_2 = \mathcal{E}_2 - (R_2 + R_3) I_3 - I_1 R_2 = 0 \qquad (4)$$

$$\Rightarrow \quad I_1 = \frac{\mathcal{E}_2}{R_2} - \frac{(R_2 + R_3)}{R_2} I_3 \qquad (5)$$

Put (5),(3) into (1)

$$\mathcal{E}_1 - I_1 R_1 - I_1 R_2 - I_3 R_2 = \mathcal{E}_1 - (R_1 + R_2)\left[\frac{\mathcal{E}_2}{R_2} - \frac{(R_2 + R_3)}{R_2} I_3\right] - I_3 R_2 = 0$$

$$\Rightarrow \quad \frac{(R_1 + R_2)(R_2 + R_3)}{R_2} I_3 - I_3 R_2 = \mathcal{E}_2 \frac{R_1 + R_2}{R_2} - \mathcal{E}_1 \qquad (6)$$

$$\Rightarrow \quad \frac{R_1 R_2 + R_1 R_3 + R_2 R_3}{R_2} I_3 = \mathcal{E}_2 \frac{R_1 + R_2}{R_2} - \mathcal{E}_1 \qquad (7)$$

$$\Rightarrow \quad I_3 = \frac{R_2}{R_1 R_2 + R_1 R_3 + R_2 R_3}\left[\mathcal{E}_2 \frac{R_1 + R_2}{R_2} - \mathcal{E}_1\right]$$

$$= \frac{1}{R_1 R_2 + R_1 R_3 + R_2 R_3}\left[\mathcal{E}_2 (R_1 + R_2) - \mathcal{E}_1 R_1\right] \qquad (8)$$

Put (8) into (5)

$$I_1 = \frac{\varepsilon_2}{R_2} - \frac{R_2+R_3}{R_2}\,\frac{1}{R_1R_2+R_1R_3+R_2R_3}\left[\varepsilon_2(R_1+R_2)-\varepsilon_1 R_2\right]$$

$$= \frac{\varepsilon_1(R_2+R_3)}{R_1R_2+R_1R_3+R_2R_3} + \frac{\varepsilon_2(R_1R_2+R_1R_3+R_2R_3)-\varepsilon_2(R_1+R_2)(R_2+R_3)}{R_2(R_1R_2+R_1R_3+R_2R_3)}$$

$$= \frac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_1 - \frac{R_2^2}{R_2(R_1R_2+R_1R_3+R_2R_3)}\,\varepsilon_2$$

$$= \frac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_1 - \frac{R_2}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_2 \qquad (9)$$

Put (8) & (9) into (3)

$$I_2 = \frac{R_2+R_3}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_1 - \frac{R_2}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_2 + \frac{(R_1+R_2)}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_2 - \frac{R_2}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_1$$

$$= \frac{R_3}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_1 + \frac{R_1}{R_1R_2+R_1R_3+R_2R_3}\,\varepsilon_2 \qquad (10)$$

Since R_2 is small, (9) & (8) show that I_1, I_3 are positive, and are hence in the direction drawn. I_2 is also positive and in the direction drawn.

5 points for messy algebra

SOLUTIONS

4(a) $$V = \int dV = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \int_0^a \frac{1}{4\pi\varepsilon_0}\frac{dr' \cdot 2\pi r'\sigma}{\sqrt{(r')^2+z^2}}$$

(7 points) (+3 points)

$$\left(dq = (2\pi r' dr')\,\sigma\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot 2\pi\sigma\left(\sqrt{(r')^2+z^2}\right)\Big|_0^a$$

$$= \boxed{\frac{\sigma}{2\varepsilon_0}\left(\sqrt{a^2+z^2} - z\right)}$$

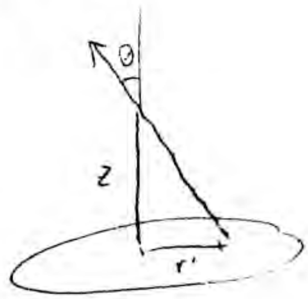
$$= 2k\pi\sigma\left(\sqrt{a^2+z^2} - z\right)$$

(7 points) Easy Way

(b) $$E_z = -\frac{\partial V}{\partial z} = -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{a^2+z^2}} - 1\right)$$

$\left(\text{only 4 points if used } -\frac{\partial V}{\partial a}\right)$

$$= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)\hat{z}$$ upward



Difficult Way

By symmetry, there is only a field in the z-direction.

$$E = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\cos\theta$$

$$= \int_0^a \frac{1}{4\pi\varepsilon_0}\frac{(2\pi r' dr')\,\sigma}{(r')^2+z^2}\cdot\frac{z}{\sqrt{(r')^2+z^2}} = -\frac{z\sigma}{2\varepsilon_0}\left((r')^2+z^2\right)^{-1/2}\Big|_0^a$$

[−3 for forgetting the $\cos\theta$]

$$= \frac{z\sigma}{2\varepsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{z^2+a^2}}\right)$$

$$= \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+a^2}}\right)$$

upward

Use superposition!

(c)

(6 points)

(+2 pts for using superposition)

$$E_{\text{infinite plane}} = \frac{\sigma}{2\epsilon_0} \quad (+2 \text{ pts})$$

$$\therefore \vec{E} = \left(\frac{\sigma}{2\varepsilon_0} - \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{a^2+z^2}}\right)\right)\hat{z}$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\frac{z}{\sqrt{z^2+a^2}}\hat{z}$$

(+2 points)

(d)

(5 points)

$$\vec{F} = -q\vec{E} = -\frac{q\sigma}{2\varepsilon_0}\frac{z}{\sqrt{z^2+a^2}} \approx -\frac{q\sigma}{2\varepsilon_0 a}z$$

Restoring force is proportional to displacement $\Downarrow$ SHO

$$m\frac{d^2z}{dt^2} = -\frac{q\sigma}{2\varepsilon_0 a}z \Rightarrow z = A\sin(\omega t + \phi)$$

where $\therefore \omega = \sqrt{\dfrac{+\frac{q\sigma}{2\varepsilon_0 a}}{m}}$

$$\omega = \sqrt{\frac{+q\sigma}{2m\varepsilon_0 a}}$$

(-2 points if answer involves z)