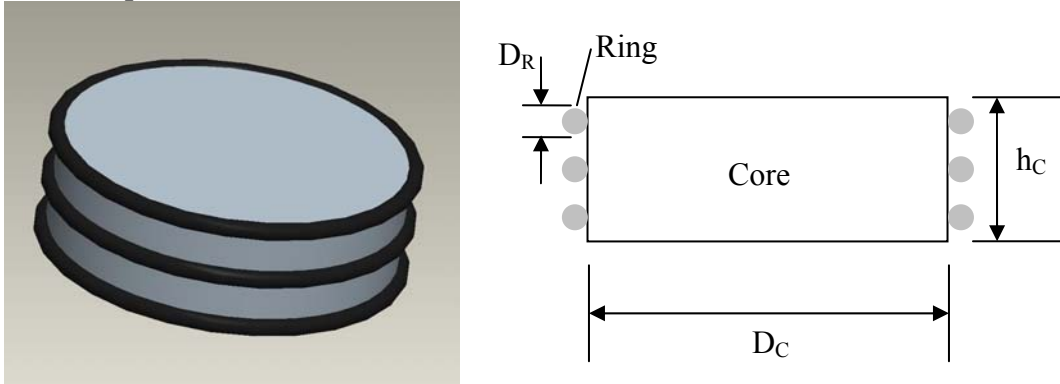This is a closed notes/closed book exam. Show all work. Write your name on every page.

NAME:

_____

(100 pts) You are an engineer working for a medical device company, and a surgeon proposes a design for a spinal disc replacement that is shown below.

Figure 1. Spinal Disc Replacement (Not to scale)

The device is made up of a core that is surrounded by rings. The surgeon tells you that the reasoning behind their design is to mimic the basic structure of the natural disc. The core supports load by pressurization, and the rings will keep the core from over expanding. This is analogous to the nucleus pulposus and annulus fibrosus of the natural intervertebral disc.

The core is made from a soft and incompressible material, and the rings are made of Cobalt-Chrome wire. This initial design has 3 total rings. Dimensions and material properties of the device are given below:
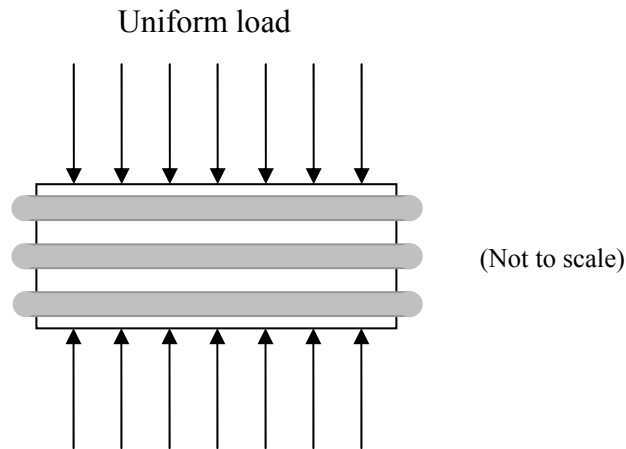
Core
- Incompressible
- Diameter, $D_C = 3.00$ cm
- Height, $h_C = 1.00$ cm

Ring
- Modulus, $E = 200$ GPa
- Yield Strength, $\sigma_{yld} = 600$ MPa
- Diameter, $D_R = 0.50$ mm
- 3 total rings (<u>initial</u> design)

Your job is to determine if this is a safe and feasible design. Consider the uniform axial loading scenario shown below.

Uniform load

(Not to scale)

Figure 2. Illustration of Uniform Uniaxial Loading

Static Analysis

1. *Calculate the stress in the rings for a worst-case load equal to 4X body weight. Assume 70 kg for mass.*

   a. *Calculate the pressure in the disc. Since this is an incompressible material, the pressure will be the same in ALL directions in the core.*

   The pressure, p, in the disc is the axial force divided by the axial cross-sectional area.

   Answer 1a →
   $$p = \frac{Force}{Area_{AXIAL}} = \frac{4 \times BW}{\frac{\pi}{4}D_C^{\,2}} = \frac{(4)(70kg)\left(9.8\,m/s^2\right)}{\frac{\pi}{4}(0.03m)^2} \Rightarrow 3.9 MPa \qquad \textbf{(1)}$$

   b. *Perform equilibrium on the following cut section of the device to determine the hoop stress, σ_{hoop}, in the rings. The force due to the pressure must be balanced by the force in the rings. Assume uniform stress, σ_{hoop}, in the rings. (Note: Solve for σ_{hoop} symbolically because you will be using that equation again.)*

Hoop stress, σ_{hoop} (out of page)

y

z

x

Pressure (all directions)

(Not to scale)
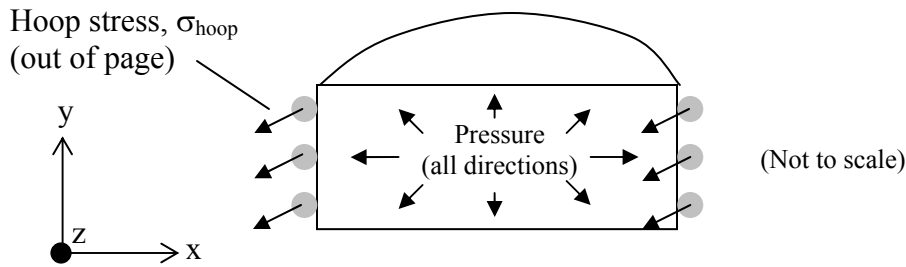
Figure 3. Section of Device for Equilibrium Analysis

The method to calculate the hoop stress in the rings is similar to that used to derive the hoop stress for a cylindrical thin walled pressure vessel.

Static equilibrium is enforced to calculate the hoop stress.

$$\sum F_Z = 0 \qquad (2)$$

Assume the z-axis is out of the page and is positive. Note that the only net force, due to pressure, is in the z-direction. The x and y components for pressure cancel out. The only net forces are shown below. The force due to pressure, F_p, is balanced by the force due to hoop stress, F_{HOOP}.
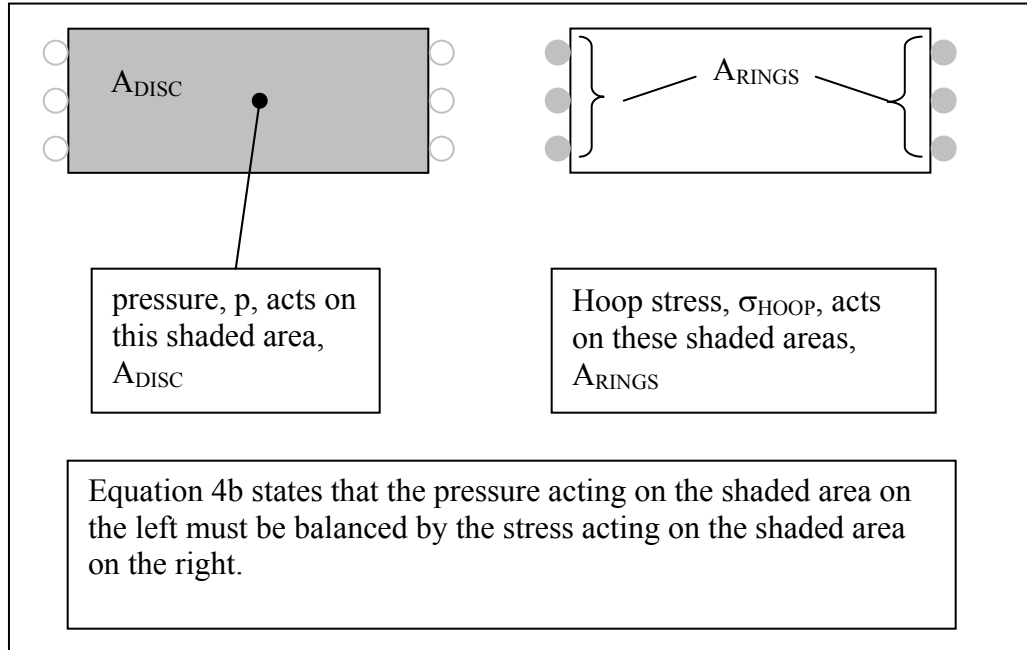
$$-F_p + F_{HOOP} = 0 \qquad (3)$$

Fp, is equal to the pressure multiplied by the projected area, and F_{HOOP} is equal to the hoop stress multiplied by the total area of the rings.

$$-pA_{DISC} + \sigma_{HOOP}A_{RINGS} = 0 \qquad (4a)$$

$$pA_{DISC} = \sigma_{HOOP}A_{RINGS} \qquad (4b)$$

Equation 4b is illustrated in the next figure.

The area of the disc is given by the following equation.

$$A_{DISC} = D_C h_C \qquad\qquad (5)$$

It will be convenient to put the area of the rings in terms of an arbitrary number of rings, N.

$$A_{RINGS} = \frac{\pi}{4} D_R^{\,2}(2N) \qquad\qquad (6)$$

(the factor of 2 is there due to the fact that each ring has two faces on the cross section as shown in Figure 3.)

$$p D_C h_C = \sigma_{HOOP} \frac{\pi}{4} D_R^{\,2}(2N) \qquad\qquad (4c)$$

Solve for σ_{HOOP}

Answer 1b $\longrightarrow$
$$\sigma_{HOOP} = \frac{2 p D_C h_C}{\pi D_R^{\,2} N} \qquad\qquad (4d)$$

(For 3 rings as shown in the initial design, the stress is equal to 989 MPa.)

c. ***What is the minimum number of rings needed for a safety factor for yield of the rings equal to at least 2?***

The answer from Problem 1b (Eq. 4d) can be used to solve for N.

$$N = \frac{2pD_c h_C}{\pi D_R^{\,2} \sigma_{HOOP}}$$
(4e)

The given safety factor allows you to calculate the allowable hoop stress.

$$Safety\_Factor = \frac{Allowable}{Applied} = 2 = \frac{\sigma_{YLD}}{\sigma_{HOOP}} \Rightarrow \sigma_{HOOP} = \frac{\sigma_{YLD}}{2} = \frac{600MPa}{2} = 300MPa$$

All quantities are known so you can solve for N directly using Eq. 4e.

$$N = \frac{(2)(3.9 \times 10^6 \, Pa)(0.03m)(0.01m)}{\pi (0.005m)^2 (300 \times 10^6 \, Pa)}$$

Answer 1c $\longrightarrow$ $\quad N = 9.9 \Rightarrow 10$ rings.

The number of rings must be rounded up to the next whole number.
The resulting hoop stress for 10 rings is equal to 297 MPa.

2. *The spine experiences approximately 2 million total loading cycles per year and you want the device to last 25 years. Assume that the stress cycles vary from zero stress to maximum values as follows:*
- *98% of total cycles experience stress equal to 0.5 x σ_{hoop}*
- *1.5% of total cycles experience stress equal to 0.8 x σ_{hoop}*
- *0.5% of total cycles experience stress equal to 1.0 x σ_{hoop}*
(σ_{hoop} is the value resulting from the design you calculated in 1c.)

You have actual fatigue data for the material used for the rings as shown in the S-N plot in Figure 4. The data was generated from tests performed with a stress range from 0 to the maximum value, S.

a. *What is the endurance limit of this material?*

Answer 2a $\longrightarrow$

The endurance limit is defined as the maximum cyclic stress value, S, where the material does not fail. According to the data, this material does not fail when subjected to cyclic stress values as high as 200 MPa.

b. *Will the rings survive for the required number of cycles?*

Miner's Rule must be used to determine if the rings will fail. The rings will survive if the following inequality is satisfied.

$$\sum \frac{n_i}{N_i} \le 1 \rightarrow \frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} \le 1 \tag{7}$$

The values for n can be found from the problem statement where each is a percentage of the number of total required cycles. A total of 50 million cycles are required.

$$n_1 = 2.5 \times 10^5$$
$$n_2 = 7.5 \times 10^5$$
$$n_3 = 49 \times 10^6$$

The values for N must be found from the S-N plot. A best fit line can be used to determine N values for each value of S.

$$S_1 = 1.0 \times \sigma_{HOOP} = 297 MPa$$
$$S_2 = 0.8 \times \sigma_{HOOP} = 237 MPa$$
$$S_3 = 0.5 \times \sigma_{HOOP} = 148 MPa$$

The values for N corresponding to the values for S are drawn on the S-N plot. Approximate N values are given below.

$$N_1 = 3.5 \times 10^5$$
$$N_2 = 1.0 \times 10^6$$
$$N_3 \rightarrow \infty$$

Values for n and N can be plugged in to Equation 7.

Answer 2b $\longrightarrow$ $$\frac{2.5 \times 10^5}{3.5 \times 10^5} + \frac{7.5 \times 10^5}{1.0 \times 10^6} + \frac{49 \times 10^6}{\infty} \approx 1.5 \ge 1 \Rightarrow FAIL$$

A design with 10 rings will not satisfy fatigue requirements. Since reading values from the S-N plot is subjective, it is more important for you to understand the procedure rather than get the exact numbers shown here.

c. ***What is the minimum number of rings necessary for the part to survive the required number of cycles?***

There are many ways to approach this problem. One method is to observe that adding just enough rings to drop S_2 below the endurance limit will likely result in an acceptable design. Values for stress can be calculated

using Eq. 4d.

Stress	Rings (N)		
	10	**11**	**12**
S$_1$	297	269	247
S$_2$	237	215	198
S$_3$	148	134	124

New values for N can be found from the S-N plot and plugged into Equation 7.

Answer 2c $\longrightarrow$
$$\frac{2.5\times10^5}{9\times10^5}+\frac{7.5\times10^5}{\infty}+\frac{49\times10^6}{\infty}\approx0.3\leq1\Rightarrow PASS$$

The resulting minimum number of rings is approximately 12. However, since reading N values from the S-N plot is subjective, the technique is more important than the final number of rings.
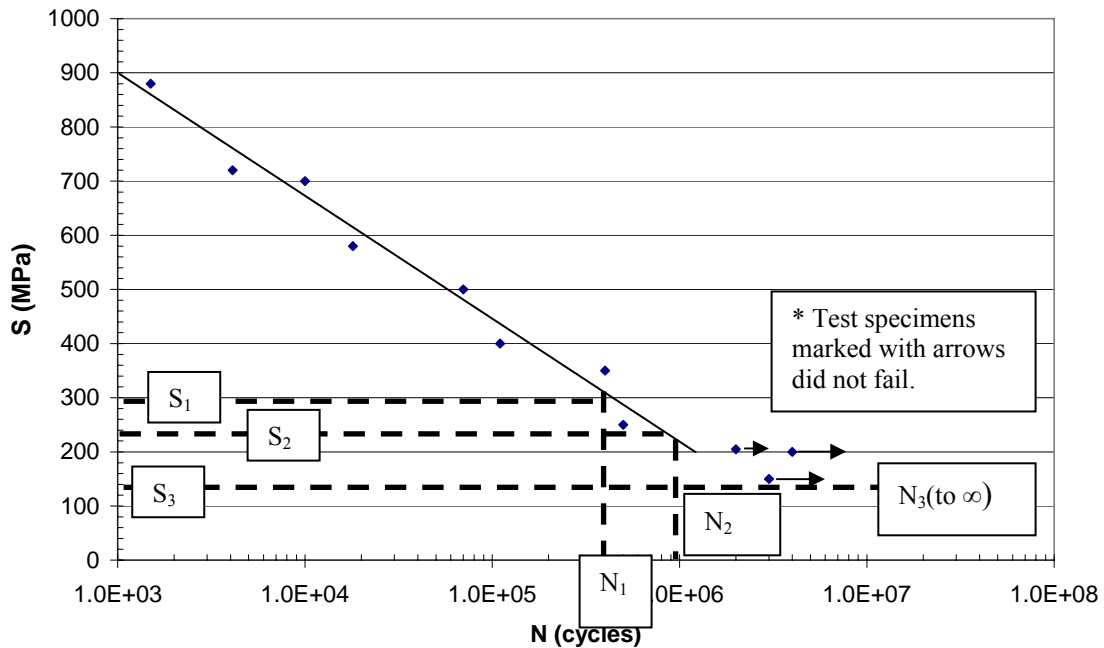
S-N Data for Ring Material

Figure 4. S-N Data

3. *During implantation trials, it is seen that occasionally the surgeon makes a scratch on one of the wires, forming a surface flaw. This appears to be unavoidable, so you must design the part to tolerate a certain initial flaw size caused after quality control inspection.*
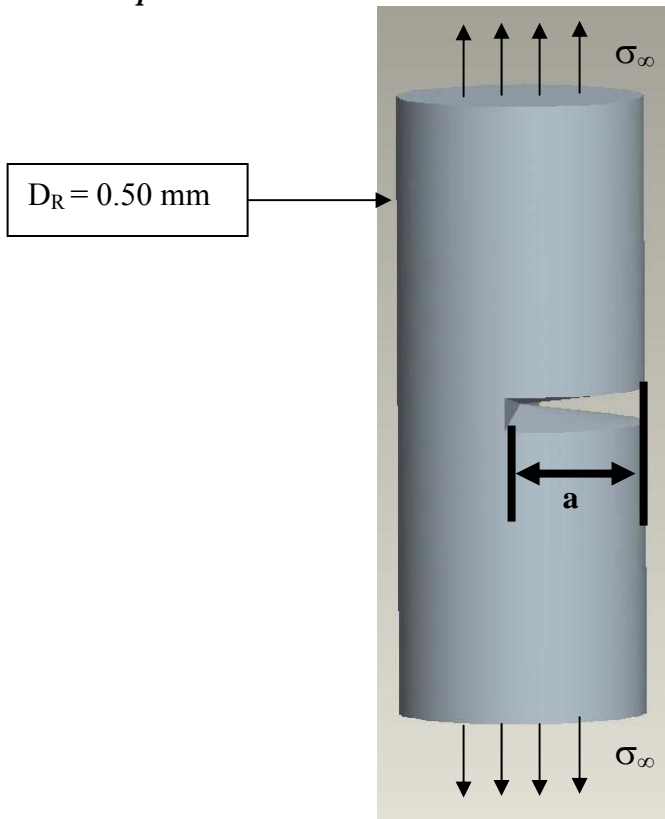

Figure 5. Flawed Wire

- *Critical stress intensity (for this specimen size) is 90 MPa*m$^{1/2}$*
- *"Paris" coefficients are C=9e-10 and m=2.96.(for da/dN in mm/cycle, K in MPa*m$^{1/2}$)*
- *Threshold stress intensity (this case) is 5 MPa* m$^{1/2}$.*

a. *Given the equation for the stress intensity factor:*

$$K_I = Y_{(a)}\sigma^\infty \sqrt{\pi a}$$

where

    Y(a)= 1.2 if 0<a<0.1mm
    Y(a)= 1.5 if 0.1<a<0.4 mm
    Y(a)=2.0 if 0.4<a<0.5mm.

Separate and integrate the Paris Equation to find the number of cycles required to grow a crack from an initial size, a_i, to a final size a_f. Assume here that 0.1<a<0.4mm. Show symbolically. Ignore any possible threshold effects.

We start with the Paris equation,

$$\frac{da}{dN} = C\Delta K^m$$

where we know that K is a function of the crack length a, so we collect stuff that is a function of a onto the left side, and dN onto the right side.

$$K_I = Y_{(a)}\sigma^{\infty}\sqrt{\pi a}$$

$$\frac{da}{C\Delta K^m} = dN$$

$$\frac{da}{C\left[Y_{(a)}\left(\Delta\sigma^{\infty}\right)\sqrt{\pi a}\right]^m} = dN$$

Thus, we integrate from the intitial crack state to the final one.

$$\int_{a_i}^{a_f}\frac{da}{C\left[Y_{(a)}\left(\Delta\sigma^{\infty}\right)\sqrt{\pi a}\right]^m} = \int_{N_i}^{N_f}dN$$

The problem states that we are considering Y(a) as a constant, since we are in only one of the piecewise defined regimes by presumption. Therefore we can pull out all constant terms from the integrals.

$$\frac{1}{C\left[Y\left(\Delta\sigma^{\infty}\right)\sqrt{\pi}\right]^m}\int_{a_i}^{a_f}\frac{da}{\sqrt{a}^m} = \int_{N_i}^{N_f}dN$$

Thus, we need only integrate $a^{-m/2}$ between the limits and the rest is just numerical.

$$\int_{a_i}^{a_f}\frac{da}{\sqrt{a}^m} = \int_{a_i}^{a_f}a^{-\left(\frac{m}{2}\right)}da = \left(\frac{1}{-\frac{m}{2}+1}\right)a^{\left(-\frac{m}{2}+1\right)}\Bigg|_{a_i}^{a_f}$$

Thus, we have

$$\int_{N_i}^{N_f} dN = N = \frac{1}{C\left[Y\left(\Delta\sigma^\infty\right)\sqrt{\pi}\right]^m}\left(\frac{1}{-\frac{m}{2}+1}\right)a^{\left(-\frac{m}{2}+1\right)}\Bigg|_{a_i}^{a_f}$$

which is the symbolic form of the number of cycles required to propagate a crack between two specified crack lengths.

     b.

> ***i. Only if you solved 3(a): Numerically calculate result from 3(a) above, with all parameters as given above in this problem and a_i=0.11mm and a_f=0.25mm, except leave the applied stress as a variable. This gives the life as a function of the applied stress. Ignore threshold and critical effects.***

Calculate coefficient (with C for m/cy for simplicity):

$$\frac{1}{C\left[Y\left(\Delta\sigma^\infty\right)\sqrt{\pi}\right]^m} = \frac{1}{(9e-13)\left[1.5\sqrt{\pi}\right]^{2.96}}\left(\sigma^\infty\right)^{-2.96} = 6.148e10\left(\sigma^\infty\right)^{-2.96} = Q$$

Units here are ambiguous. Now,

$$N = Q\left(\frac{1}{-\frac{m}{2}+1}\right)a^{\left(-\frac{m}{2}+1\right)}\Bigg|_{a_i}^{a_f} = Q\left(\frac{1}{-\frac{2.96}{2}+1}\right)\left(a_f^{\left(-\frac{2.96}{2}+1\right)} - a_i^{\left(-\frac{2.96}{2}+1\right)}\right) = 6.148e10\left(\sigma^\infty\right)^{-2.96}(-2.08)\left(a_f^{-0.48} - a_i^{-0.48}\right)$$

Thus, for a_i=1.1e-4 m and a_f=2.5e-4 m

$$N = 3.31e12\left(\sigma^\infty\right)^{-2.96} cycles,$$ where the stress must be input in MPa!

> ***ii. Only if you did not solve 3(a): Estimate the life of the part by computing da/dN for five equally spaced crack lengths between a_i=0.11mm and a_f=0.25mm, and assuming that da/dN is constant between these crack length states, like in the homework. Find the estimated life as a function of only the stress, as described in 3(b_i). Ignore threshold and critical effects.***

Crack lengths: a_1=0.11mm, a_2=0.145mm, a_3=0.18mm, a_4=0.215mm, a_5=0.25mm. (Δa=0.035mm)

$$\frac{da}{dN} \approx \frac{\Delta a}{\Delta N} \approx C(\Delta K)^m = C\left[Y_{(a)}\left(\Delta\sigma^\infty\right)\sqrt{\pi a}\right]^m = (9e-13)\left[1.5\sqrt{\pi a}\right]^{2.96}\left(\sigma^\infty\right)^{2.96}$$

Thus, we rearrange

$$\frac{\Delta a}{(9e-13)\left[1.5\sqrt{\pi a}\right]^{2.96}(\sigma^\infty)^{2.96}} = \Delta N$$

Thus for a1 to a2 (using average crack length)

$$\Delta N_{12} = \frac{0.035e-3}{(9e-13)\left(1.5\sqrt{\pi(0.128e-3)}\right)^{2.96}}(\sigma^\infty)^{-2.96} = 1.25e12(\sigma^\infty)^{-2.96}$$

$$\Delta N_{23} = 8.72\,e11\,(\sigma^\infty)^{-2.96}, \Delta N_{34} = 6.54\,e11\,(\sigma^\infty)^{-2.96}, \Delta N_{45} = 5.76\,e11\,(\sigma^\infty)^{-2.96}$$

Thus, adding all these increments up we get

$$\Delta N_{15} = N = 6.54e11(\sigma^\infty)^{-2.96} = 3.35e12(\sigma^\infty)^{-2.96}$$

This is a very good estimate (when compared to the closed form solution)!

c. ***Find the applied stress required for the crack to take 50 million cycles to propagate in the analysis in part 3(b). How many reinforcement rings does this require?***

We must remember (at all times) that the Paris equation in this case uses stress in MPa. If you put a stress of 7MPa into it as 7e6 Pa, you will get out nonsense.

Using the answer from above, we have N=50e6 cycles:
$$N = 3.3e12(\sigma^\infty)^{-2.96}$$

$$\sigma^\infty(MPa) = \left(\frac{50e6}{3.3e12}\right)^{\frac{-1}{2.96}} = 42.5MPa \text{ (for N 50 million cycles).}$$

Thus for our life, we need the rings to experience no more than 42.5 MPa in service for a safety factor of one, or for failure right at 50 million cycles. We are not considering safety factor in this problem, which may be a dubious concept here anyway.

From the solution to the first problem we have the number of rings required for a given hoop stress on the ring. Here we assume that force on the implant is the nominal weight of the patient. Other assumptions for service force may also be applicable, if so stated.

$$N_{rings} = \frac{hMg}{2\sigma\left(\frac{\pi}{4}\right)^2 Dd^2} = \frac{(0.01m)(70kg)9.8\left(m/s^2\right)}{2\sigma\left(\frac{\pi}{4}\right)^2(0.03m)(0.0005m)^2} = \frac{7.41e8\,N/m^2}{\sigma}$$

For stress 42.5e6 (N/m^2) we require N$_{rings}$>17.4 rings, or 18 discrete rings for implant to survive 50 million cycles in this condition. Note that this is near maximum, as only 20 can be stacked within the unloaded height of the implant (1cm), and there will be some compression of the device in service. This may require a design iteration!

> **d. Calculate the critical crack length for the stress computed in 3(c). Will the crack reach this length in service? If so, when? If not, when will failure occur?**

We are given that
$$K_c = 90 MPa\sqrt{m}$$

and we know
$$K_I = Y_{(a)}\sigma^\infty \sqrt{\pi a}$$

so we back out the crack length that gives this stress intensity under the stress computed above.

$$\left(\frac{90 MPa\sqrt{m}}{1.5(42.5 MPa)}\right)^2 \frac{1}{\pi} = a_c = 0.63\text{m (!) This uses a first guess of Y=1.5 for a<0.4mm.}$$

$$\left(\frac{90 MPa\sqrt{m}}{2(42.5 MPa)}\right)^2 \frac{1}{\pi} = a_c = 0.36\text{m (!) Since } a_c>0.4\text{mm, but this has little meaning.}$$

This is very large, and thus we do not ever expect fast fracture in the component.

Incidentally, the critical crack size for the yield stress is 1.8 mm, which is still much larger than the wire. Also, for an ultimate stress of 800 MPa, the critical crack size is 1 mm, both assuming Y=2.

Since the crack will not spontaneously propagate to failure, then it seems that the crack must propagate through the entire wire thickness for it to break. This is not the case. As the crack propagates, the cross sectional area reduces, and hence the stress over the section ahead of the crack increases. When the stress throughout this "critical" section reaches the ultimate strength of the material, the wire snaps. This happens when about half of the thickness of the wire is compromised, and hence why a$_f$=0.25mm was chosen.