Saturday, December 14, 12:30–3:30 PM, 2002.

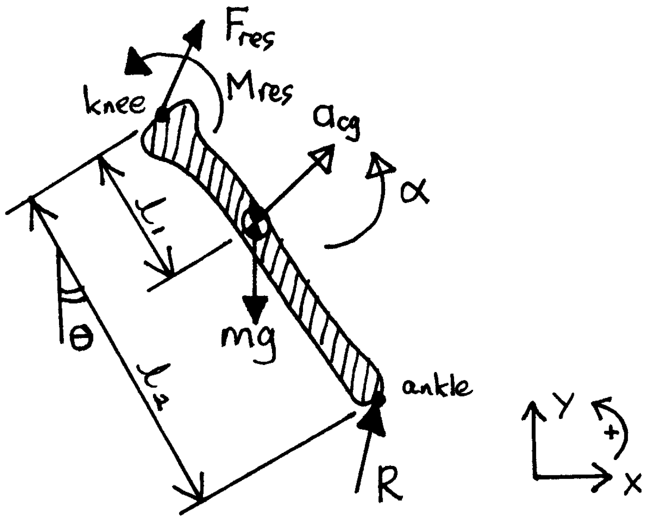
Answer all questions for a maximum of 100 points. Please write all answers in the space provided. If you need additional space, write on the back sides. Indicate your answer as clearly as possible for each question. Write your name at the top of each page as indicated. *Read each question very carefully!*

---

1. Dynamic Analysis and Gait [15 points total]

The sketch below shows a free-body diagram of an idealization of the shank during gait at an angle θ to the vertical. For simplicity, the foot has been ignored, and the ground reaction force $\boldsymbol{R}$ is assumed to act at the ankle. Also shown are the resultant force $\boldsymbol{F_{res}}$ and moment $\boldsymbol{M_{res}}$ acting at the knee joint.

Express the resultant moment $\boldsymbol{M_{res}}$ in terms of the ground reaction force $\boldsymbol{R}$, the mass of the shank m, the moment of inertia of the shank with respect to its center of gravity I_{cg}, and the accelerations of the shank ($\boldsymbol{a_{cg}}$ and α). All unknowns have been drawn in their assumed positive directions. Express all vectors in X-Y components.



2. Beam on Elastic Foundation Theory [20 points total]

Derive the following equations for bending of a beam on an elastic foundation. State clearly *all* the assumptions.

i) $$\frac{\partial V}{\partial x} = -p + k\square$$

ii) $$\frac{\partial M}{\partial x} = -V$$

iii) $$\frac{\partial}{\partial x^2}\left\{EI\,\frac{\partial^2\square}{\partial x^2}\right\} + k\square = p$$

where x is the distance along the length of the beam, V is the shear force acting on the beam, M is the bending moment acting on the beam, p is a uniformly distributed force/length acting on the beam, k is the foundation modulus, EI is the flexural stiffness of the beam with respect to its neutral axis, and $\square = \square(x)$ is the displacement of the neutral axis of the beam.

Hints: • Start with an equilibrium analysis of an element of the beam of length ∂x.

• For bending of a beam with "small" deformations of the neutral axis $\square$, the following approximation holds where $\square$ is the radius of curvature of the deformed neutral axis:

$$\frac{1}{\square} \;\square\; \frac{\partial^2\square}{\partial x^2}$$

(iv) Starting with $\square(x)$, describe how you would calculate the interface stresses acting between the beam and the foundation.

(v) In the context of beam-on-elastic-foundation theory, explain briefly the concept of "flexible" and "rigid" behaviors and distinguish them from "compliant" and "stiff" behaviors.

(vi) For a single beam of modulus *E*, moment of inertia *I*, length *L*, resting on an elastic foundation of foundation modulus *k*, "rigid" behavior is promoted by increasing which parameters?

3. Composite Beam Theory, Implant Design, and Biological Heterogeneity [20 points total]

(i) One important concept in hip stem design is that the design specifications of the prosthesis can depend as much on the patient characteristics, *i.e*. bone dimensions, than on the stem itself. Consider an uncemented hip stem in the mid-diaphysis. Assume that the stem and bone comprise two concentric circular cylinders in cross-section, with a stem diameter D_s (which equals the inside bone diameter for this press-fit uncemented device) and an outside bone diameter of D_b . You have two tasks:

a. The stem (alone) is tested pre-implantation in pure bending and is found to reach the end of its elastic range at a bending moment M_{safe}. Write out an expression for the corresponding maximum elastic bending stress on the stem surface, σ_{safe}.

b. Say now that you want to design such that the maximum elastic bending stress in the stem is <u>one half</u> of σ_{safe} when the device is implanted in the bone. Assume here that the stem-bone system behaves as a composite beam subjected to a bending moment of M_{safe}, Show that this design will depend on the size of the bone (*i.e*. the outside bone diameter). Specifically, derive the following:

$$D_s = D_b \left(\frac{E_b}{E_b + E_s} \right)^{\frac{1}{4}}$$

where E_s and E_b are the Young's moduli of the stem and bone, respectively.

(ii) Provide an interpretation of this equation.

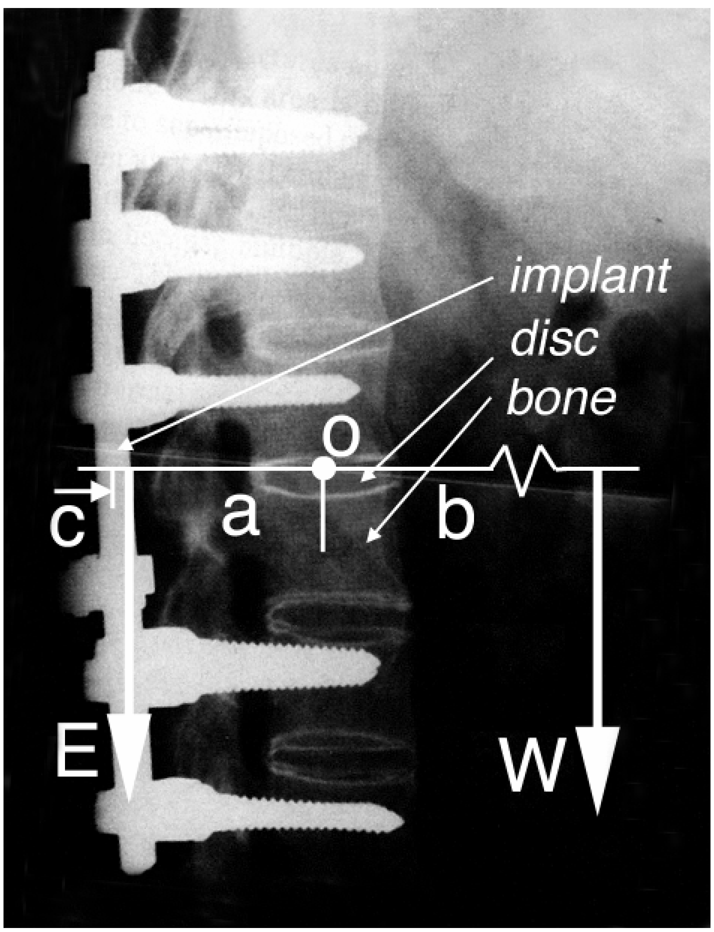
(iii) For a given outside bone diameter, will the stem diameter that meets the above condition be larger or smaller (in diameter) if CoCr alloy is used for the stem instead of Ti-6Al-4V alloy? Explain.

4. Design of Spine Implants [20 points total]

The picture below shows an X-ray of a pedicle screw fixation type of implant commonly used in the spine. Under normal conditions (no implant or pathology), the *erector spinae* muscle force E counteracts the gravity force W, and the spinal joint contact force acts through the bone at point o (for simplicity, all forces are assumed to act only in the vertical direction). The distances a and b show the moment arms of these forces about point o.

Assume now that a disc is destroyed and cannot transfer any load between its adjacent vertebrae. A vertical implant, placed posterior to the vertebra, is therefore used to transmit loads between the vertebrae, and is held in place using horizontal screws. Assume that the neutral axis of the (symmetrical) implant is at a distance c posterior to the action of the muscle force E, as shown. Further assume that the body is accustomed to activating the muscles at their normal values for any given activity and that this will not be changed upon implantation of the prosthesis.

As a design engineer, you have been asked to specify the minimum width of the implant such that its maximum allowable elastic bending stress is no greater than 350 MPa when W = 350 N (ignore any effects of axial stresses). Assume that the thickness of the rectangular-shaped cross-section (into the page) is 24 mm. Use the following dimensions: a = 50 mm; b = 150 mm; c = 5 mm.



5. Viscoelasticity and Tissue Mechanics [25 points total]

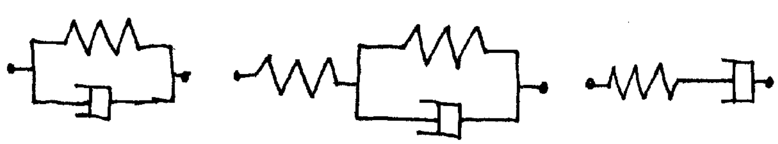
(i) Describe, with diagrams, three manifestations of viscoelastic behavior:

A.

B.

C.

(ii) Match the spring-dashpot models with their creep responses (a stress was applied, then removed after steady-state):



(iii) We studied the theory of *linear* viscoelasticity in this course. What does the "linear" refer to?

(iv) Derive, including any relevant diagrams, and explain the meaning of the “hereditary integral”. Start with a discrete (summation) version and progress to the continuous (integral) version.

(v) This graph shows a comparison of the creep response to the same static load for a healthy vs. severely degenerated intervertebral disc. Give two interpretations of this graph.