ME 176 Final Exam, Fall 1995 *Name:*____

Saturday, December 16, 12:30–3:30 PM, 1995.

Answer all questions. Please write all answers in the space provided. If you need additional space, write on the back sides. Indicate your answer as clearly as possible for each question. Write your name at the top of each page as indicated. *Read each question very carefully!*

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1. (20 points total) Forces and Moments at Joints

A. [10 points] Draw a fully labeled free-body diagram of the forearm during a curl exercise (with a 20 kg mass in the hand) that could be used to calculate the *resultant* force and moment acting on the elbow joint. Include all accelerations in your free body diagram and ignore the mass of the hand.

B. [10 points] Based on your free-body diagram, write out the moment equation of motion (dynamics equation) *about the elbow joint*. Assume that the elbow joint is fixed in space. Regarding moment of inertia terms, assume only the value for the forearm about its centroid (*i.e*. this is the only moment of inertia term that should appear explicitly in your equation).

2. (25 points total) Bone Mechanics and Beam Theory

A. [10 points] When a material such as cortical bone is loaded in pure bending until its collapse, the neutral axis of bending shifts as the material yields due to the asymmetry of yield strengths of the bone material. Derive the expression shown here for the distance from the compressive surface of the beam to the neutral axis, $\hat{y}$, in terms of the depth of the beam *d*, and the yield strengths in tension σ^t and compression σ^c:

$$\hat{y} = \frac{d\,\sigma^t}{\sigma^t + \sigma^c}$$

Assume elastic-perfectly-plastic (Figure 1) behavior of the beam material. The cross-section of the beam is rectangular, of width *w* and depth *d*, and the beam is loaded in pure bending.

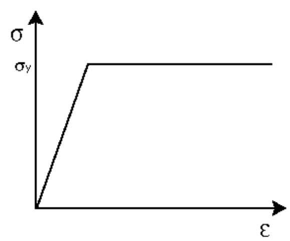

Figure 1: Elastic-perfectly-plastic behavior of a material, showing the yield stress σ_y. Yield stresses are different in tension and compression.

B. **[5 points]** When loaded to failure in a four-point bend test (*i.e.* when a pure bending moment is applied to the beam), the bone specimen will fail at the so-called “fully plastic moment” M_p , which is given by the following expression:

$$M_p = \frac{w\ d^2\ \sigma^t\ \sigma^c}{2\ (\sigma^t + \sigma^c)}$$

Assuming w = 3 mm and d = 1 mm, calculate a value for M_p using typical values of the tensile and compressive strengths (in the longitudinal direction) of human cortical bone.

C. **[10 points]** Calculate the percentage error in the tensile yield strength measurement that occurs with respect to the correct value if one assumes that the compressive strength is equal to the tensile strength.

3. (25 points total) Design and Analysis of Hip Prostheses

A. [5 points] Write out an expression for the resultant bending moment M at section X-X for the two-dimensional model of a cemented bone-prosthesis system shown in Figure 2. Assume that the joint contact force J and the abductor force A act at angles α and β, respectively, as shown. The (x, y) coordinates of the points a (the femoral head center), b (the assumed single attachment point of the abductors), and c (the center of the cross-section at X–X) are (a_x, a_y), (b_x, b_y), and (c_x, c_y), respectively.

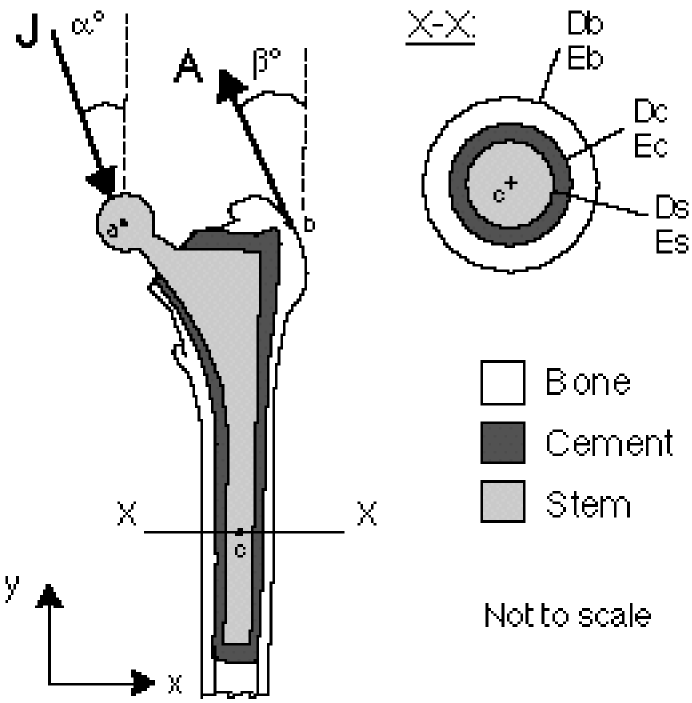

Figure 2: Two-dimensional model of a cemented titanium-alloy hip prosthesis, showing the cross-section at the mid diaphysis (D = outside diameter; E = Young's modulus; subscripts *b*, *c*, and *s* refer to the bone, cement, and stem, respectively).

B. **[5 points]** Assuming ideal load sharing between the bone, stem, and cement, write out the expression for the tensile bending stress on the lateral surface of the cemented stem at the cross-section X–X of Figure 2. Include explicit expressions for all second moment of area terms (*i.e.* give these parameters in terms of diameters etc.).

C. **[10 points]** Starting from the composite beam theory equation used in part (B) above, derive the following equation for the diameter of the cemented stem D_s^* — the so called "critical" stem diameter — that maximizes the bending stress in the stem as the stem diameter is varied and everything else is held constant:

$$D_s^* = \left[\frac{64\, E_b I_b}{3\pi E_s}\right]^{\frac{1}{4}}$$

To simplify the math, at the start of this problem assume that the flexural stiffness of the cement is negligible compared to that of the stem and bone, *i.e.* $E_c I_c << E_b I_b$ and $E_s I_s$.

D. **[5 points]** Discuss any clinical implications of the expression in part (C) in the context of prosthesis selection for a patient with the bone geometry as shown in Figure 2.

4. (30 points total) Design of Knee Prostheses

A. [6 points] For the classical Hertz contact problem, where are the locations of the:

(i) maximum compressive stress?

(ii) maximum tensile stress?

(iii) maximum shear stress?

What are the implications of these stress locations for damage mechanisms in the plastic component of a total knee prosthesis?

B. [15 points] Derive the following equations for bending of a beam on an elastic foundation. State clearly *all* the assumptions.

i) $$\frac{\partial V}{\partial x} = -p + k\square$$

ii) $$\frac{\partial M}{\partial x} = -V$$

iii) $$\frac{\partial}{\partial x^2}\left\{EI\,\frac{\partial^2\square}{\partial x^2}\right\} + k\square = p$$

where *x* is the distance along the length of the beam, *V* is the shear force acting on the beam, *M* is the bending moment acting on the beam, *p* is a uniformly distributed force/length acting on the beam, *k* is the foundation modulus, *EI* is the flexural stiffness of the beam with respect to its neutral axis, and $\square = \square(x)$ is the displacement of the neutral axis of the beam.

Hints: • Start with an equilibrium analysis of an element of the beam of length *∂x.*

• For bending of a beam with "small" deformations of the neutral axis $\square$, the following approximation holds where $\square$ is the radius of curvature of the deformed neutral axis:

$$\frac{1}{\square} \; \square \; \frac{\partial^2\square}{\partial x^2}$$

Question 4B continued:

C. **[5 points]** For a beam of length *L* which is supported by an elastic foundation and centrally loaded by a compressive force *P*,

$$\text{plot a graph of } \frac{\square_{end}}{\square_{center}} \text{ vs. } \square L,$$

where $\square = \left(\frac{k}{4EI}\right)^{\frac{1}{4}}$ is a dimensionless parameter, and $\square_{end}$ and $\square_{center}$ are the displacements of the beam end and center, respectively. Use a range of $\square L$ values from 0–6. Based on this graph, identify which of the parameters *k*, *E*, *I*, and *L*, when increased, promote "stiff" behavior of the beam.

D. **[4 points]** Indicate if the maximum contact stress in the artificial knee joint (Figure 3) should increase or decrease if:

a) R_f is increased (all else constant)

b) R_t is increased (all else constant)

c) *t* is decreased (all else constant)

d) the modulus of the plastic is increased (all else constant)

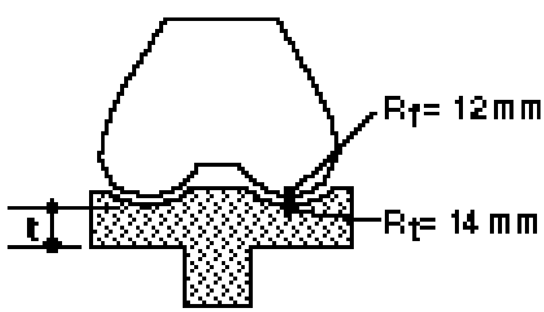

Figure 3: Frontal view of a bicondylar total knee prosthesis. R_f, R_t are the radii of the metal femoral and plastic tibial components, respectively; *t* is the average thickness of the plastic tibial component.

Additional work space for miscellaneous questions (indicate question):