ME 176 Final Exam, Fall 1997 *Name:*____

Tuesday, December 16, 5:00–8:00 PM, 1997.

Answer all questions for a maximum of 100 points. Please write all answers in the space provided. If you need additional space, write on the back sides. Indicate your answer as clearly as possible for each question. Write your name at the top of each page as indicated. *Read each question very carefully!*

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1. (30 points total) Dynamic Planar (2D) Analysis of the Skeleton

During a "giant circle" routine (Figure 1), a gymnast rotates in the sagittal plane around a fixed bar which they grip in an approximately frictionless manner. In some routines, the gymnast will actively move their legs from the hyperextended position (shown below) to a more flexed position, *i.e.* they will move their legs closer to the front of the body, thereby decreasing the angle between the anterior aspect of the torso and the legs. As with most kinetic problems in musculoskeletal biomechanics, we can use the inverse-dynamics approach to solve for the internal loads in the body, where some of the motions and external loads are directly measured and the unknown internal loads are then solved for with the appropriate dynamics analysis.

In analyzing this situation, assume that the body can be modeled as a two-link rigid bar system (Figure 1b), one rigid bar (termed here the "torso") representing the torso, head, arms, and hands, and the other rigid link (termed here the "legs") representing the legs and feet. The links are connected at the hip joint. The masses and mass moments of inertia (about the mass center) are m_T, I_T for the torso, and m_L, I_L for the legs. Assume for this problem that the only known kinematic data are the angular position, velocity, and accelerations of the torso, denoted θ_T, ω_T and α_T, respectively. The resultant (vector) force exerted by the bar on the gymnast's hands, **G**, is also known. The appropriate dimensions are known and are given in Figure 1.

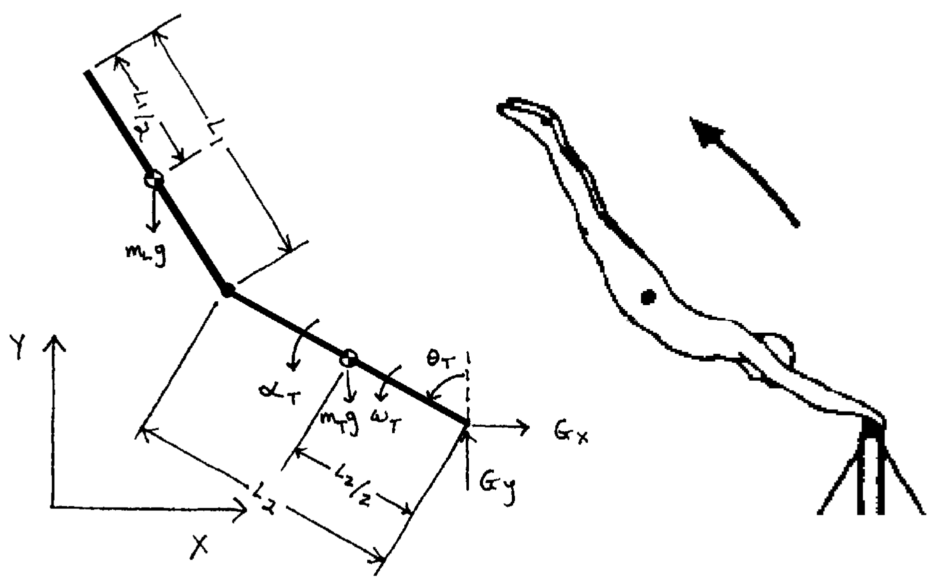

Figure 1: Two-bar linkage model (left) of gymnast performing a "giant circle" routine (right).

A. Draw a fully labeled free-body diagram for each link. Add also the appropriate linear and angular accelerations.

B. What is the relationship, if any, between the linear and angular accelerations for: (i) the torso; and (ii) the legs? Explain your answers very briefly.

C. By analyzing the torso, derive an expression for the resultant moment exerted on the hip joint by the legs during such a routine in terms of *only the known kinematic and geometric data for the torso*.

D. What are the remaining unknowns in this problem and how many independent equations remain? Based on this information, briefly outline the analysis required, including the need to measure any quantities directly if necessary, in order to solve for these unknowns.

2. (30 points total) Osteoporosis and Composite Beam Theory

With a fall to the side of the hip, a medially directed force F at the greater trochanter can generate a force J on the femoral head at an angle □ to the femoral axis (as shown in Figure 2) and some distal equilibrating loads (not shown). The angle between the neck and femoral axis is □. Of particular interest is the importance of the eccentricity of the trabecular bone with respect to the cortical bone within the femoral neck, *i.e.* the circular cross-section of trabecular bone is not centered within the circular cross-section of cortical bone in the femoral neck.

□. Derive the following expression for the position of the neutral axis $\hat{y}$ for the cross-section shown in Figure 2:

$$\hat{y} = \frac{c(1 \square \square)}{1 + \square(\square^2 \square 1)}$$

where □ is the ratio of the cortical to trabecular diameters, D_{cort}/D_{trab}; □ is the ratio of cortical modulus to trabecular modulus, E_{cort}/E_{trab}; and c is distance between the center of the cortical and trabecular areas.

B. Write out the expression for the maximum tensile stress on the inferior aspect of section A–A for the loading conditions shown in Figure 2 in terms of the appropriate forces, dimensions, and material properties, the parameter $\hat{y}$, and the areal moments of inertia for the trabecular I_{trab} and cortical I_{cort} bone with respect to their centroids. Assume that composite beam theory applies, and neglect analysis of shear stresses.

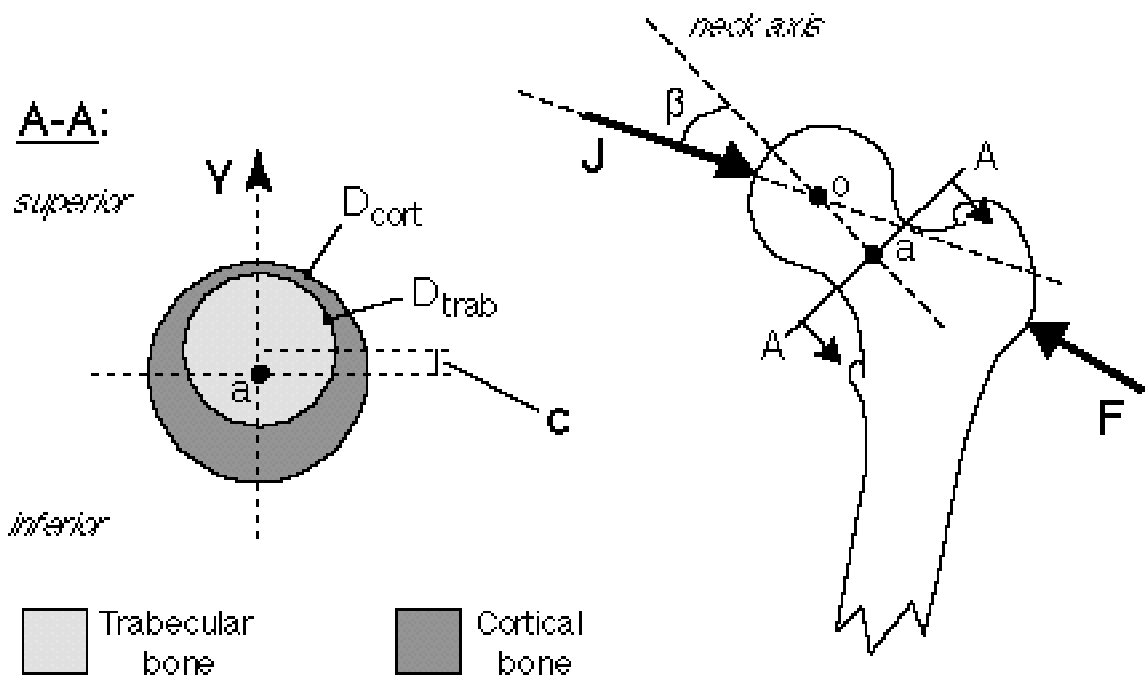

Figure 2: Left: Cross-section of the femoral neck at section A–A, showing the eccentricity of the trabecular bone with respect to the cortical bone and the correspondingly thicker cortical shell on the inferior aspect of the neck. a is the position of the center of the cortical bone area at the section. The distance between the centroids of the cortical and trabecular bone is c, and the diameter of each type of bone is D_{cort} and D_{trab}, respectively, with corresponding moduli of E_{cort} and E_{trab}. **Right:** Model of the proximal femur when impacted at the greater trochanter. In response to the external trochanteric contact force F, an internal joint contact reaction force J is generated at the hip acting towards the center of the femoral head o as shown (making an angle □ to the femoral axis). The equilibrium forces and moments that must act distally are not shown.

C. In the study of osteoporosis, there is often confusion about the terms used to describe bone density. Use Archimedes' Principle to derive an expression for the tissue density of trabecular bone, in terms of the weight of the bulk specimen and the weight of the specimen while submerged in water (assume the density of water is 1 gm/cm^3).

3. Bone Remodeling (15 points)

Cortical bone gets slightly weaker with aging, but it turns out that diaphyseal diameters can also change with aging, particularly in males, such that the safety factor of the diaphysis is approximately constant with aging. To illustrate how subtle changes in the periosteal dimensions can offset loss of tissue strength, solve the following problem.

Assume that in order to offset age-related reductions in the strength of cortical bone tissue, the safety-factor *SF* (defined as the failure stress divided by the functional stress) for the bone tissue on the periosteal surface of the diaphysis remains constant:

i.e. $SF_1 = SF_2$ at the periosteal surface,

where the subscript *1* refers to the young bone before any remodeling changes or loss of tissue strength, and subscript *2* refers to the old bone after remodeling and loss of tissue strength. [Consider only tensile stresses.] If the tissue tensile strength of cortical bone is decreased to 80% of its original ("young") value with aging, what is the change in periosteal diameter required to keep $SF_1 = SF_2$ at the periosteal surface?

Assume for simplicity the following:
(i) the diaphysis can be modeled as a hollow circular cylindrical tube;
(ii) the diaphysis is loaded by a pure bending moment *M*, and stresses can be calculated from simple beam theory;
(iii) the endosteal diameter is constant and equal to 15 mm;
(iv) the initial ("young") value of the periosteal diameter is 30 mm.

4. (25 points total) Miscellaneous

A. [5 points] Fill in the following table of material properties with approximate values (full marks here for 10 correct entries and no mistakes):

Material	*Young's modulus (MPa)*	*Compressive Strength (MPa)*	*Tensile Fatigue Strength (MPa)*
Cortical bone †			
Trabecular bone ††			
UHMWPE			—
PMMA		—	
Tendon		—	—
Ti-6Al-4V alloy		—	
CoCr alloy		—	
316L stainless steel		—	

† for cortical bone, give values for longitudinal loading (for human bone)
†† for trabecular bone, give a mean value typical of the elderly human spine
— do not give any values

B. [5 points] Indicate if the maximum contact stress in the artificial knee joint (Figure 4.1) would increase or decrease if:

a) R_f were increased (all else constant).
b) R_t were increased (all else constant).
c) t were increased (all else constant).
d) the modulus of the plastic were increased (all else constant).

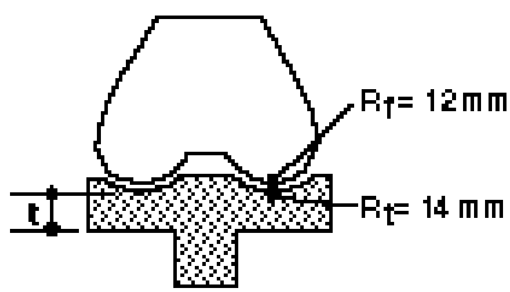

Figure 4.1: R_f, R_t are the radii of the metal femoral (white) and plastic tibial (shaded) components, respectively; t is the average thickness of the plastic tibial component.

C. [5 points] Sketch a typical shear stress distribution along the interface of two concentric cylinders, loaded in compression where all load is transferred through the interface from one cylinder to the other. Assume that the overall structural behavior of the cylinders is "flexible", as in the context of beam on elastic foundation theory, and that the axial stiffness (EA) is larger for the inside component. How would this distribution change if the modulus of the inside component was decreased so its axial stiffness was less than that of the outside component?

D. **[10 points]** The general time-dependent behavior of many soft tissues, and even bone, can be modeled approximately using spring-dashpot models, such as in Figure 4.2.

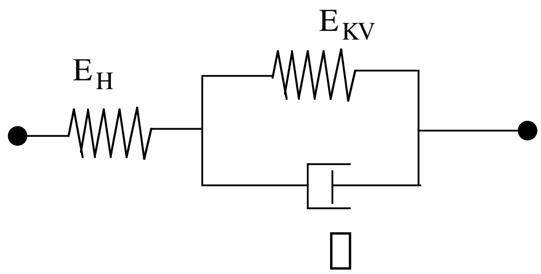

Figure 4.2: "Standard Linear Solid" viscoelastic spring-dashpot model.

Show that the equation relating stress σ to strain ε for this model is of the form:

$$\frac{\partial \sigma}{\partial t} + a\,\sigma = b\,\frac{\partial \varepsilon}{\partial t} + c\,\varepsilon$$

by deriving expressions for the constants *a*, *b*, and *c* in terms of E_H, E_{KV}, and η.

Additional work space (indicate clearly which question):