Tuesday, December 15, 5:00–8:00 PM, 1998.

Answer all questions for a maximum of 100 points. Please write all answers in the space provided. If you need more space, there is an additional page at the end. Indicate your answer as clearly as possible for each question. Write your name at the top of each page as indicated. *Read each question very carefully!*

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1. (15 points total) Dynamic Planar (2D) Analysis of the Skeleton

Gravity is the most common force used to offer resistance in physical therapy. An exercise mass m_{ex} is placed on a patient's foot for quadriceps strengthening (Figure 1). This mass behaves as a concentrated mass acting at a distance L from the knee joint center o. The patient, sitting on a chair, is asked to extend their shank off the floor. At some instant, the shank is at an angle θ to the vertical, as shown, and can be assumed to be rotating about the fixed knee center o with angular velocity ω and angular acceleration α. Assume that: 1) the mass of the shank and foot (not including the exercise mass) is m_{leg} with a center of mass at a distance L_1 from the knee center and a mass moment of inertia $\bar{I}_{leg}$ about this point; 2) the ankle joint is rigid; 3) the only other loads acting on the shank are the joint resultant force and moment ($F_{res\text{-}k}$ and $M_{res\text{-}k}$ respectively) at the knee joint; and 4) this is a two-dimensional planar rigid body problem.

(i) [7 points] Draw a free body diagram, including all accelerations, of the knee/foot/mass system.

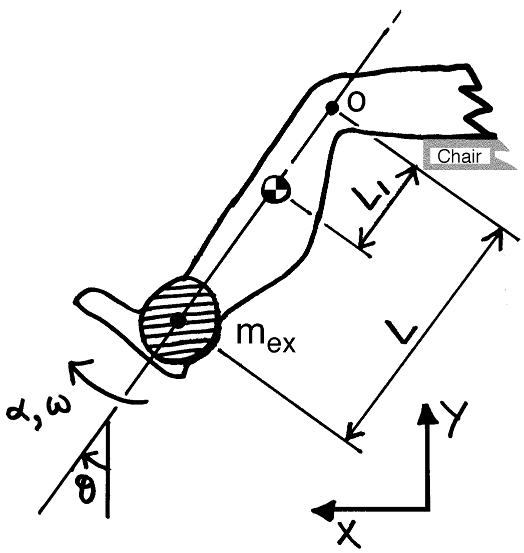

Figure 1

(ii) [8 points] Write out the equation of rotational motion about the knee joint center. Express all inertial parameters explicitly in terms of $\bar{I}_{leg}$, m_{ex} and m_{leg}.

2. (25 points total) Shear Lag Theory, Load Sharing, and Load Transfer

(i) [10 points] For shear-lag theory applied to two concentric cylinders loaded axially:

(a) draw the free-body diagrams of the relevant differential elements of the structure

(b) write out the equations for the kinematic assumptions

(c) write out the equations for the constitutive assumptions

(d) write out the boundary conditions

(ii) [5 points] From these equations, derive the overall governing differential equation:

$$\frac{d^2 \square}{dx^2} - \left[\frac{\square\,(1 + \square)\,\bar{D}\,G}{t\,E_1\,A_1} \right] \square = 0$$

where □ is the shear strain in the interface material, x is the distance along the interface, □ is the ratio of the axial stiffnesses (E_1A_1/E_2A_2), $\bar{D}$ is the average diameter of the interface, G is the shear modulus of the interface, and the subscripts refer to the axially loaded cylinders (1 is the outer cylinder, 2 is the inner cylinder).

(iii) [6 points] Sketch a typical graph of shear stress along the interface, assuming "flexible" behavior of the structure. Identify the load sharing and load transfer regions.

(iv) [4 points] What is the condition that optimizes the strength of the interface?

3. (10 points total) Bone Biomechanics

The vertebral body is often modeled as two concentric elliptical cylinders, an outer shell of cortical bone and a trabecular bone core. Assume such a model shown (Figure 2) where the compressive force *P* is applied centrally and distributed uniformly via a rigid plate (a common *in vitro* mechanical testing protocol).

For that model, expression the uniform strain in the vertebral body as a function of the applied force *P*, the major and minor cortical diameters (D_1 and D_2, respectively), the moduli E_1 and E_2, and the cortical thickness, *t*.

Note: the area of an ellipse is $\pi D_1 D_2/4$

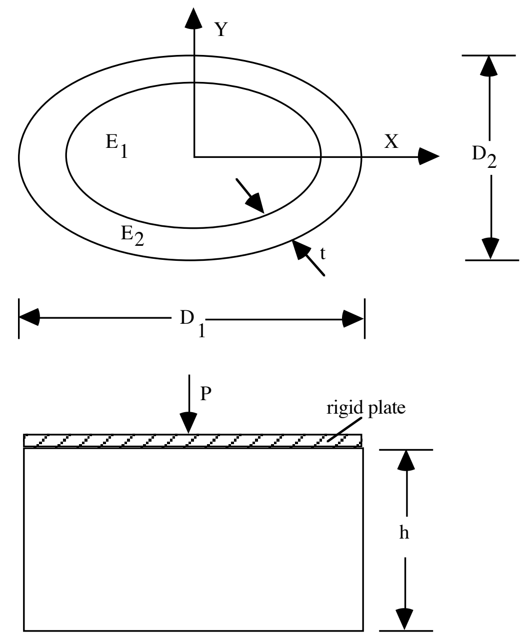

Figure 2: Simple elliptical model of the vertebral body. Top: cross-section; bottom: vertical section.

4. (30 points total) Composite Beam Theory and Hip Prosthesis Design

In one of your previous course assignments, you were asked to optimize the design (modulus and stem diameter) of a femoral stem. The chosen optimization criterion was to minimize bone resorption. Thus, the bone stresses which resulted from the prosthesis-cement-bone composite(Figure 3A) should be similar in magnitude to naturally occurring bone stresses at the diaphysis of the femur (Figure 3C). This criterion was subjected to a number of constraints, the most important of which were:

1) No bone should be removed. Thus, the bone endosteal diameter must equal the cement outer diameter. Since there are no gaps at the bone-prosthesis interface, the inner diameter of the cement equals the stem diameter (Figure 3A). To maintain structural integrity of the cement, its *thickness* must be greater than or equal to 0.5 mm.

2) Assuming that both the neck and stem portions of the prosthesis can be modeled as circular cylinders of identical diameter (Figure 4), the flexural stiffness of the stem (alone) should equal the flexural stiffness of the femoral neck (Figure 3B). This is to ensure that the overall stability is sufficient, *i.e.* the prosthesis doesn't deform too much upon load bearing. Table 1 gives all the relevant dimensions for the problem.

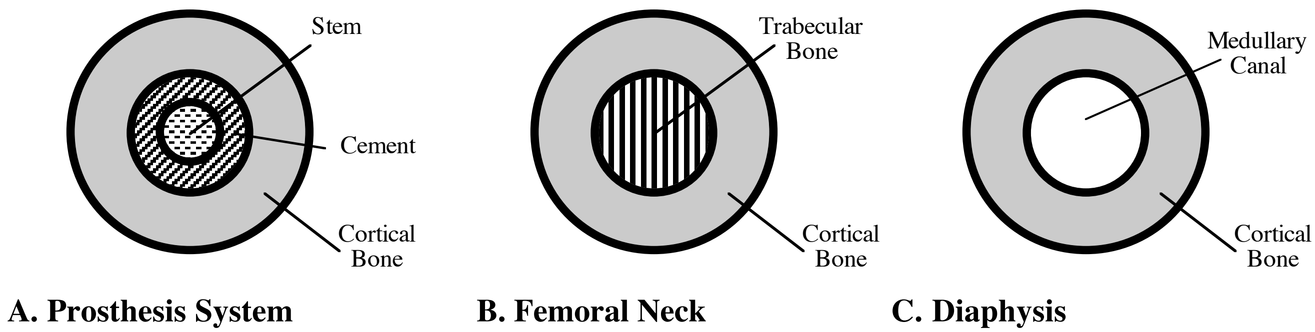

Figure 3: Schematics of the cross-sections of: A) the prosthesis-cement-bone composite; B) the femoral neck; and C) the diaphysis of the femur.

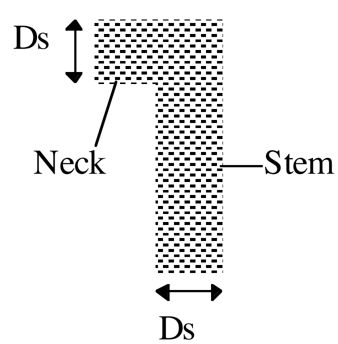

Figure 4: Side view of the idealized hip prosthesis. Both the neck and stem regions of the prosthesis are modeled as circular cylinders with identical diameter.

Table 1: Bone dimensions (mm) for two different bones.

	Periosteal Diameter	Endosteal Diameter	Neck Outer Diameter	Neck Inner Diameter
Bone A	30.2	14.6	30.2	25.2
Bone B	43.3	28.4	43.3	38.3

(i) [4 points] The objective function for this optimization problem can be written as: max J = $\square_{bc} / \square_{ba}$ where $\square_{ba}$ and $\square_{bc}$ are the bone stress "alone" (*i.e.* no implant) and the bone stress as part of the composite, respectively. The function may be written in this manner since $\square_{ba}$ is always greater than $\square_{bc}$. Both $\square_{ba}$ and $\square_{bc}$ are the stresses at the periosteal diameter. Write out the expression for J in terms of:

Dp = periosteal diameter	Eb = modulus of cortical bone
De = endosteal diameter	Ec = modulus of cement
Ds = stem diameter	Es = modulus of stem
M = applied moment	

(ii) [8 points] The analysis for the project showed that the solution to this objective function was achieved when the largest possible stem diameter was used. Based on your expression for J, explain why this is so. [Hint: in your expression from (i), think about what is variable vs. constant or constrained.]

For the larger bone sizes (Table 1, Bone B), the optimal stem modulus was 45 GPa. However, in the project discussions, many of you stated that there were no biocompatible materials corresponding to that modulus. Instead, the use of Ti-alloy (E=110 GPa) was suggested. The following two questions seek to explore how the use of this material affects the issue of bone resorption.

(iii) [4 points] Calculate the optimal stem diameter corresponding to the use of Ti-alloy for Bone B. [Assume the following: Eb=17 GPa (cortical bone) and Et=1 GPa (femoral neck trabecular bone)]

(iv) [4 points] Calculate the ratio: $\square_{bc}$ / $\square_{ba}$ for Bone B using the Ti-alloy stem design parameters [E=110 GPa; Ds as determined from part (iii)]. In addition to the constants given in part (iii), assume M=150 Nm; $\square_{ba}$=23.1 MPa; and the cement modulus Ec=2.3 GPa.

(v) [3 points] When using its optimal design parameters [Es=45 GPa ; Ds=27.4 mm], the ratio $\square_{bc}$ / $\square_{ba}$ for Bone B equals 65.6%. The manufacturing engineers at an implant company are very interested to know if Ti-alloy can be used for a large range of bone sizes since dealing with a single material would simplify the manufacturing process and save money. Based on your calculated ratio in part (iv), is Ti-alloy a viable material option for the larger bone sizes?

(vi) [4 points] The manufacturing engineers ask you an even more basic question: Why can't a single prosthesis design (one stem diameter and one material) be implemented over the entire range of bone sizes? The optimal design parameters for Bone A were Es=225 GPa and Ds=13.6 mm. Using this particular design, calculate $\square_{bc}$ / $\square_{ba}$ for Bone B.

(vii) [3 points] Compare the ratio of part (vii) with the ratio provided in part (v) and state which is more optimal. Based on this result, <u>and any other relevant biomechanical considerations that this analysis may have overlooked</u>, provide an answer to the manufacturer's question raised in part (vi).

5. (20 points total) Miscellaneous

(i) [5 points] What are the approximate compressive and tensile strengths, in the longitudinal direction, for human cortical bone? Make sure to include the units.

Compressive: Tensile:

Fill in the following table of modulus properties (in units of MPa) with approximate values:

Material	*Young's modulus (MPa)*
Trabecular bone †	
Articular cartilage ††	
Tendon	
UHMWPE	
PMMA	
CoCr alloy — cast	
CoCr alloy — forged	
316L stainless steel	

† for trabecular bone, give a mean value typical of the elderly human spine for inferior-superior loading
†† for articular cartilage, give the equilibrium compressive modulus

(ii) [5 points] For the plastic component of an artificial knee joint, plot a typical graph of maximum contact stress in the plastic vs. thickness of the plastic.

(iii) [5 points] Plot a typical force-velocity and power-velocity curve (on the same graph) for muscle.

(iv) [5 points] What is a typical range of strength values (in Newtons) for the human lumbar spine?

Additional work space (indicate clearly which question):