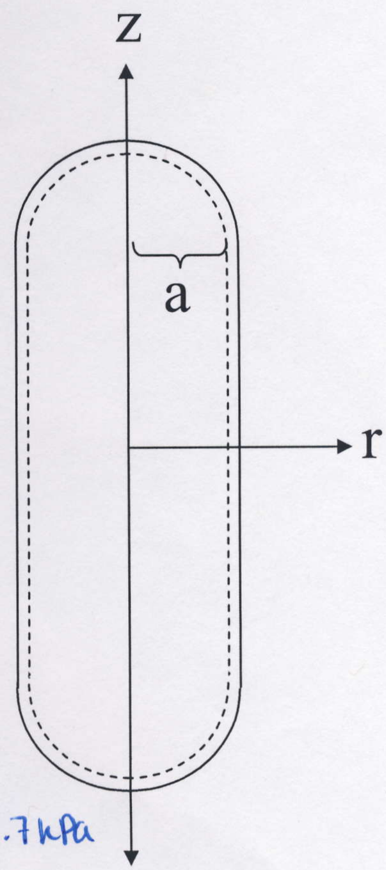
NAME: GSI solutions CAL ID:

BioE102 Midterm

Open course text, open notes, no internet or communication devices

6-8 PM

1. Consider a biomedical device which is long, cylindrical, pressurized, and capped at both ends by a hemisphere. When depressurized, the radius a is 3 cm and the thickness of the wall is 1 mm. When pressurized at 340 mmHg, the radius a is 4.2 cm and the thickness of the wall is 0.7 mm.



a. (_____ *of 5 points*) Calculate $\sigma_{\theta\theta}$ (in Pa) in the pressurized cylinder wall.

Pressurized cylinder: $a = 4.2\text{cm} = 0.042\text{m}$

$h = 0.7\text{mm} = 0.0007\text{m}$

$$h/a = \frac{0.0007\text{m}}{0.042\text{m}} < 1/20$$

So, we can use a thin wall approx.

Neglect caps for long cylinder:

$$\sigma_{\theta\theta} = \frac{Pa}{h}$$

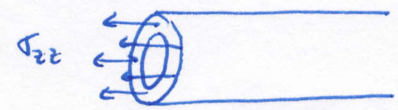
here, $P = 340\text{mmHg} \cdot \frac{100\text{kPa}}{760\text{mmHg}} = 44.7\text{kPa}$

$$\Rightarrow \sigma_{\theta\theta} = \frac{(44.7\text{kPa})(0.042\text{m})}{0.0007\text{m}} = 2682\text{ kPa}$$

$$\boxed{\sigma_{\theta\theta} = 2.68\text{ MPa}}$$

NAME: CAL ID:

b. (_____ *of 10 points*) Calculate σ_{zz} (in Pa) in the pressurized cylinder wall, and *briefly* justify your choice of the equation used to calculate it.



Assume thin walled since $h/a < 1/20$.

$$\Sigma F_z = 0$$

$$\int \sigma_{zz}\, dA - \int P_z\, dA = 0$$

$$\sigma_{zz} \int_0^{2\pi}\int_a^{a+h} r\, dr\, d\theta = \underbrace{P\int_0^{2\pi}\int_0^{a} r\, dr\, d\theta}$$

σ_{zz} constant in thin-wall approx

pressure acting over the projected area of the hemisphere on the z-face.

$$\sigma_{zz}\left\{\tfrac{1}{2}\left[(a+h)^2 - a^2\right]\right\} 2\pi = P\left(\tfrac{1}{2}a^2\right) 2\pi$$

$$\sigma_{zz}(2ah + \cancel{h^2}) = Pa^2 \quad (h^2 \sim 0)$$

$$\sigma_{zz} = \frac{Pa}{2h}$$

$$= \frac{(44.7\text{ kPa})(0.042\text{ m})}{2(0.0007\text{ m})} = 1.34\text{ MPa}$$

$$\boxed{\sigma_{zz} = 1.34\text{ MPa}}$$

(Note that this would be the same for a flat cap since the integral over the hemisphere of the pressure in the z-direction is the same.)

2. Consider the device from #1. If you were going to model this device in ADINA:

   a. (_____ *of 5 points*) What symmetry assumption can you make to simplify the geometry? Sketch your model in the simplest form that will produce an answer identical to a simulation of the device as shown.

Solution Problem 2:

A) Refer to figure 1

* Simplest model → 2D

* Symmetry → 4 identical pieces (1/4 of the cross sectional structure)

* Right Sketch → of 1/4 of the cross sectional view of the hollow structure

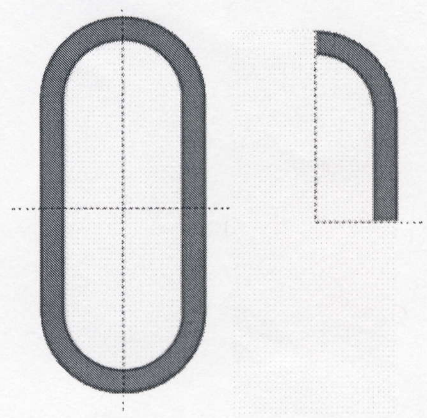


Figure 1. 1/4 of the cross sectional view of the hollow

   b. (_____ *of 5 points*) Why would you want to limit the number of elements in your model?

B)

* Reduce time/space on model/run/ analysis

c. (_____ *of 5 points*) Based on your answer to part c, explain how you would mesh this structure in the most efficient way, while still capturing the detail of the important regions. (You can draw a simple sketch and explain it. **hint:** Consider defining different surfaces)

C) Refer to figure 2

* Keep 2D approach

* Division of right 2D sketch into 2 or more surfaces

* Mesh of the surfaces made above

* Explain: Finer mesh on S_2 will capture the details of curved are. Meshing on S_1 not as detailed sue to simple geometry.

* Right sketch with meshes (optional)

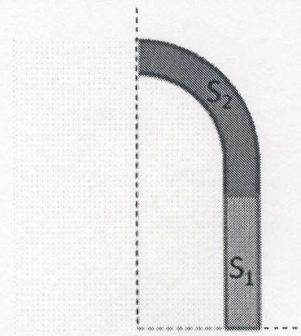
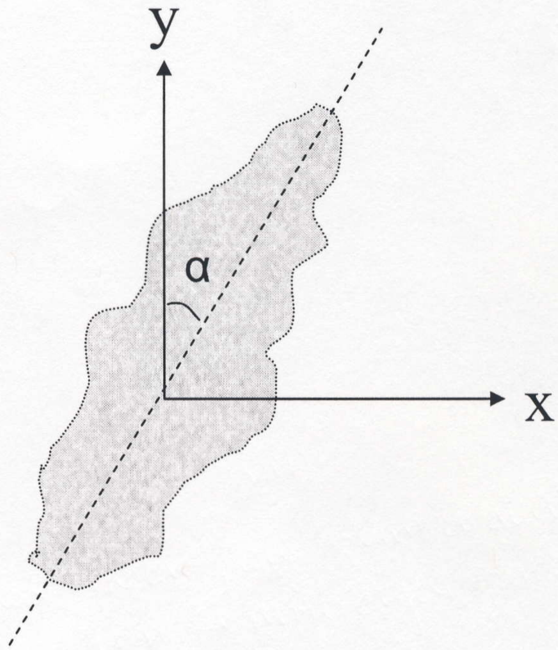
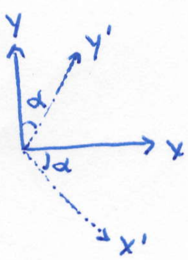


Figure2. Mesh on S_2 should be finer than mesh on S_1

3. You are studying a cell in a tissue – for simplicity's sake, the tissue is assumed to be 2D. For the given coordinate system, the state of stress in the tissue can be characterized by σ_{xx} = 170 kPa, σ_{yy} = 30 kPa, and σ_{xy} = 45 kPa. The object in grey is a cell residing in the tissue, and the dotted line indicates the line that the major axis of the cell is oriented with – simply, the way the cell is "pointing". α = 30°.



   a. (_____ *of 8 points*) Transform the coordinate system so that the new y (y') axis is oriented with the cell's major axis and calculate σ'_{xx}, σ'_{yy}, and σ'_{xy}.



$\alpha = -30°$

$$\sigma_{yy}' = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{yy}-\sigma_{xx}}{2}\cos 2\alpha - \sigma_{xy}\sin 2\alpha$$

$$= \frac{170+30}{2} + \frac{30-170}{2}\cos(-60°) - 45\sin(-60°)$$

$$= \boxed{104\text{ kPa}}$$

$$\sigma_{xx}' = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\alpha + \sigma_{xy}\sin 2\alpha$$

$$= \frac{170+30}{2} + \frac{170-30}{2}\cos(-60°) + 45\sin(-60°)$$

$$= \boxed{96\text{ kPa}}$$

$$\sigma_{xy}' = \frac{\sigma_{yy}-\sigma_{xx}}{2}\sin 2\alpha + \sigma_{xy}\cos 2\alpha$$

$$= \frac{30-170}{2}\sin(-60°) + 45\cos(-60°)$$

$$= \boxed{83\text{ kPa}}$$

b. (_____ *of 12 points*) Does the orientation of the cell line up with one of the principle stresses or the maximum ~~strain~~ Shear? If not, which is it closest to?

Angle where principal stresses occur:

$$\alpha_p = \frac{1}{2}\tan^{-1}\left(\frac{2\sigma_{xy}}{\sigma_{xx}-\sigma_{yy}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2(45)}{170-30}\right) = \underline{16^\circ}$$

Angle where max shear stresses occur:

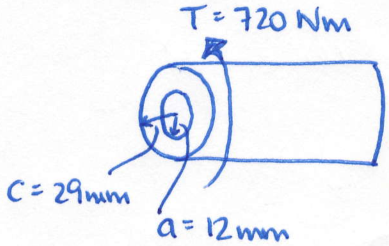
$$\alpha_s = \frac{1}{2}\tan^{-1}\left(\frac{\sigma_{yy}-\sigma_{xx}}{2\sigma_{xy}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{30-170}{2(45)}\right) = \underline{-28.6^\circ}$$

Our cell is oriented at $\alpha = -30^\circ$. This is closest to the direction of max shear ($\alpha_s = -28.6^\circ$)

4. A hollow cylindrical bone of inner radius 12 mm, an outer radius 29 mm, and a length of 15 cm is fixed at one end and exposed to a torque of 720 N∘m.

   a. (_____ *of 5 points*) Calculate the max shear stress $\sigma_{z\theta}$ (in Pa) in the bone.



$$J = \frac{\pi}{2}\left[c^4 - a^4\right] = \frac{\pi}{2}\left[(29\times10^{-3}\,m)^4 - (12\times10^{-3}\,m)^4\right]$$

$$= 1.07\times10^{-6}\,m^4$$

$$\sigma_{z\theta} = \frac{Tc}{J} = \frac{(720\,Nm)(0.029m)}{(1.07\times10^{-6}\,m^4)} = 19.3\text{ MPa}$$

$$\sigma'_{z\theta\,max} = \pm\frac{Tc}{J} = \pm\sigma_{z\theta} = \boxed{\pm 19.3\text{ MPa}}$$

b. (_____ *of 5 points*) Calculate the max shear stress $\sigma_{z\theta}$ (in Pa) in the bone if it were not hollow.

J would change for a solid bone:

$$J = \frac{\pi}{2} c^4 \qquad (\text{since } a = 0)$$

$$= \frac{\pi}{2}(0.029\,m)^4 = 1.11 \times 10^{-6}\,m^4$$

$$\Rightarrow \sigma'_{z\theta\,max} = \frac{Tc}{J} = \frac{(720\,Nm)(0.029m)}{(1.11\times10^{-6}\,m^4)} = \boxed{18.8 \times 10^6\ MPa}$$

c. (_____ *of 5 points*) What is the ratio of shear stress $\sigma_{z\theta}$ to principle stress σ_1 at any point in the hollow bone?

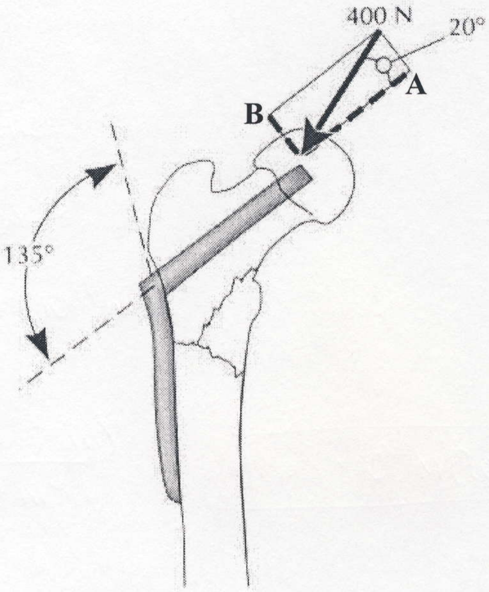
$$\sigma_{z\theta} = \frac{Tr}{J} \qquad \sigma_1 = \frac{Tr}{J}$$

$$\boxed{\frac{\sigma_{z\theta}}{\sigma_1} = 1}$$

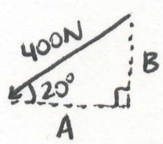
$\Rightarrow$ They are equal $\rightarrow$ 1:1 ratio

5. Consider the nail plate shown at right, which is commonly used to fix unstable intertrochanteric fractures of the femur.

   You are trying to design a nail plate for a patient with this fracture. In a static standing posture, your patient's femoral head must support a load of 400N acting at an angle of 20 degrees relative to the axis of the nail, as shown.



   a. (_____ *of 2 points*) Resolve the applied external force of 400N into components that are parallel and perpendicular to the axis of the nail (i.e. calculate forces A and B in the diagram).

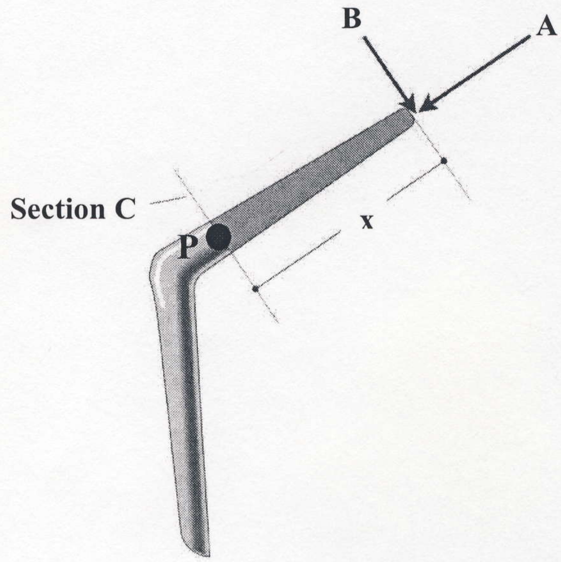


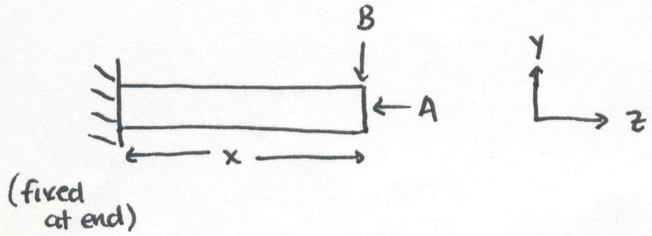
$$\text{Force}_{\parallel} = A = 400\,N \cos 20^\circ = 376\,N$$

$$\text{Force}_{\perp} = B = 400\,N \sin 20^\circ = 137\,N$$

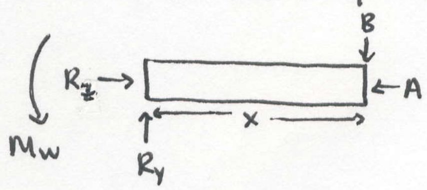
$$\boxed{\begin{array}{l} A = 376\,N \\ B = 137\,N \end{array}}$$

b. (_____ *of 9 points*) Determine the internal forces and moments that must exist in Section C of the nail, shown at right, in order to maintain equilibrium.
[Point P is considered fixed by the rest of the device.]





→ whole beam equilibrium:

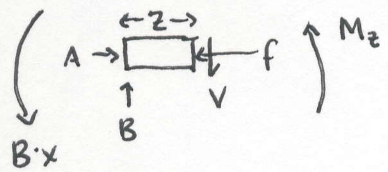


$\Sigma F_y = B - R_y = 0 \longrightarrow$ $R_y = B = 137\,N$

$\Sigma F_z = A - R = 0 \longrightarrow$ $R_z = A = 376\,N$

$\Sigma M)_p = -B \cdot x + M_w = 0 \rightarrow$ $M_w = B \cdot x = (137N)\,x$

→ partial beam equilibrium to find internal forces & moments:



$\Sigma F_y = -B + V = 0 \rightarrow$ $\boxed{V = B}$

$\Sigma F_z = -A + f = 0 \rightarrow$ $\boxed{f = A}$

$\Sigma M)_p = B \cdot x + M_z - V \cdot z = 0$

$\rightarrow \boxed{M_z = B(z - x)}$

c. (_____ *of 9 points*) The surgeon asks you if a nail plate that is made of stainless steel (elastic modulus = 180 GPa) with a length of 86mm (corresponding to length x in the diagram in part b) would be appropriate for this patient. She would like to limit deflection at the tip of the nail, since displacement of the fracture would impede the healing process.

Calculate the maximum deflection of the nail. You may treat the nail as a cantilever with an area moment of inertia of 19.5 mm^4.

$E = 180\ GPa$; $x = 86mm$; $I = 19.5\ mm^4$

As derived in class, deflection of a cantilever:

$$\delta = \frac{\text{moment} \cdot \text{length}^2}{3EI} = \frac{(F \cdot L) \cdot L^2}{3EI}$$

$$= \frac{(137N)(86\ mm)^3}{3(180\ GPa)(19.5\ mm^4)}$$

$$= \boxed{8.3\ mm \text{ deflection}}$$

Alternatively,

$$EI_{zz} \frac{d^2v}{dx^2} = M_z(x)$$

$$\int EI \frac{d^2v}{dx^2}\, dx = \int M_z(x)\, dx = \int B(x-L)$$

$$EI \frac{dv}{dx} = \frac{1}{2} Bx^2 - BLx + C_1$$

$$\int EI \frac{dv}{dx}\, dx = \int \left(\frac{1}{2} Bx^2 - BLx + C_1\right) dx$$

$$EI\, V(x) = \frac{1}{6} Bx^3 - \frac{1}{2} BLx^2 + C_1 x + C_2$$

$$V(x) = \frac{1}{EI}\left[\frac{1}{6} Bx^3 - \frac{1}{2} BLx^2 + C_1 x + C_2\right]$$

B.C: $V(0)=0$

$\frac{dv}{dx}(0) = 0$ $\Rightarrow$ $\therefore$ $C_1 = 0$

$C_2 = 0$

$$V(x) = \frac{1}{EI}\left(\frac{1}{6} Bx^3 - \frac{1}{2} BLx^2\right)$$

maximum:

$$V(L) = \frac{1}{EI}\left(\frac{1}{6} BL^3 - \frac{1}{2} BL\cdot L^2\right) = -\frac{1}{3} \frac{BL^3}{EI}$$

$= 8.3\text{mm}$. ✓

NAME: CAL ID:

6. Back in the early 17th century, Galileo postulated the existence of certain relations between the bone proportions of small animals and large animals. Specifically, he was interested in what would happen if you scaled an animal bone up. For instance, how would the proportions of the leg bones of a giant squirrel compare to the proportions of the leg bones of a normal size squirrel? We're going to examine a very simple version of this concept.

Suppose we take a femur with a periosteal radius of 2cm, an endosteal radius of 1.5cm, and a length of 25cm. We will also assume that the modulus of the bone is 17GPa.

a. (_____ *of 5 points*) Based on this information, calculate the load at which the bone will fail due to buckling. You can assume that n=1 here.

$c = 0.02m$; $a = 0.015m$; $E = 17GPa$

$$P_{cr} = \frac{\pi^2}{L^2} E I_{zz}$$

I_{zz} for a cylinder: $\frac{\pi}{4}(c^4 - a^4) = \frac{\pi}{4}([0.02m]^4 - [0.015m]^4) = 8.59 \times 10^{-8} m^4$

$$\rightarrow P_{cr} = \frac{\pi^2 (1.7 \times 10^{10} Pa)(8.59 \times 10^{-8} m^4)}{(0.25m)^2} = \boxed{2.306 \times 10^5 N}$$

b. (_____ *of 5 points*) Now, scale that bone up by a factor of 3 in all dimensions. In other words, we have an animal that is 3X larger in all respects. Recalculate the buckling load.

$c = 0.06m$; $a = 0.045m$; $L = 0.75m$; $E = 17GPa$

$$I_{zz} = \frac{\pi}{4}\left[(0.06m)^4 - (0.045m)^4\right] = 6.96 \times 10^{-6} m^4$$

$$\rightarrow P_{cr} = \frac{\pi^2 (1.7 \times 10^{10} Pa)(6.96 \times 10^{-6} m^4)}{(0.75m)^2} = \boxed{2.076 \times 10^6 N}$$

NAME:

CAL ID:

c. (_____ *of 5 points*) Taking those two critical loads, calculate the Critical Buckling Load to Volume ratio (P/V) for each case. Do smaller or larger bones have a greater P/V ratio?

→ P/V for small bone: $V_{cylinder} = \pi r^2 \cdot L$

$$V_{small} = (0.25m)(0.02m - 0.015m)^2 \pi = 1.96 \times 10^{-5}\ m^3$$

$$P_{sm}/V_{sm} = \frac{2.306 \times 10^5 N}{1.96 \times 10^{-5}\ m^3} = 1.17 \times 10^{10}\ N/m^3$$

→ P/V for large bone:

$$V_{large} = (0.75m)(0.06m - 0.045m)^2 \cdot \pi = 5.3 \times 10^{-4}\ m^3$$

$$P_{lg}/V_{lg} = \frac{2.076 \times 10^6 N}{5.3 \times 10^{-4}\ m^3} = 3.9 \times 10^9\ N/m^3$$

⇒ smaller bones have a larger P/V ratio.