Wednesday, December 11, 8:00–11:00 AM, 1996.

Answer all questions for a maximum of 100 points. Please write all answers in the space provided. If you need additional space, write on the back sides. Indicate your answer as clearly as possible for each question. Write your name at the top of each page as indicated. *Read each question very carefully!*

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1. (15 points total) Forces and Moments at Joints

Flexion-extension injuries occur frequently in vehicle collisions, the so-called "whiplash" injury. In this injury, there is a sudden impact that forces the head into flexion (chin moves towards chest), causing disruption of the posterior ligaments. If the velocity of impact is too high, the ligament can rupture, leading to serious injury of the cervical spine.

A. [5 points] Figure 1 shows a simplified diagram of the head, rotating in the counter clockwise direction about the fixed point *o*, which is the center of the occipital condylar joint at the base of the head. The neck and torso are fixed, simulating a vehicle at full stop after hitting a rigid obstacle with the torso ideally restrained by a seat belt. In this situation, the unrestrained head will continue to move (*i.e*. rotate about its base) although the neck and torso are at rest.

Convert Figure 1 into a free-body diagram as follows. The (joint) contact force $\mathbf{J}$ acts at the joint center through point *o* and is purely vertical. The vertical (posterior) ligament force $\mathbf{F_v}$ acts at point *p*, a horizontal distance $\mathbf{d}$ posterior to the joint center; and the horizontal ligament force $\mathbf{F_h}$ acts through the joint center in the anterior-to-posterior direction. Besides the weight of the head $\mathbf{W}$, there are no other forces acting on this system. The center of mass of the head is at point *cg*, the velocity of this point is $\mathbf{u}$ in the anterior direction; the distance between points *o* and *cg* is $\mathbf{r}$; points *o* and *cg* lie on the vertical Y-axis.

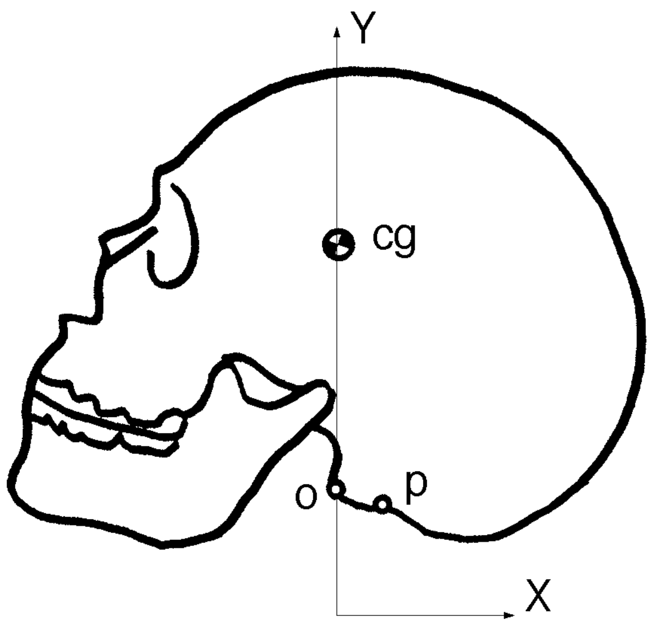

Figure 1: Schematic of the human skull, shown fixed at its base moving in pure rotation.

B. **[3 points]** Write out an expression for the angular acceleration α of the head in terms of the appropriate forces, distances, and the mass moment of inertia of the head about point *o*, I_O. Indicate also the direction of this angular acceleration.

C. **[5 points]** Show that the ratio of vertical to horizontal ligament forces is $\frac{F_v}{F_h} = \frac{I_o}{m\,d\,r}$.

D. **[2 points]** Why were no muscle forces inlcuded in this analysis?

2. (10 points total) Composite Beam Theory

Calculate the bending stress on the top surface of a beam with the cross-section shown in Figure 2. The beam is subjected to a bending moment of 100 Nm that causes tension on top, compression on bottom.

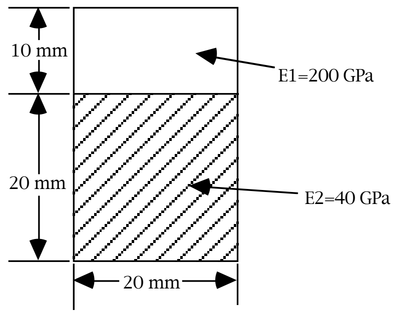

Figure 2: Cross-section of a composite beam made of two materials (E1 and E2).

3. (15 points total) Design and Analysis of Hip Prostheses

In a worst case scenario, the designer must assume that a cemented stem will become loose proximally. In that case, it is important to ensure that stem fracture will not occur.

A. [10 points] Write out an expression for the tensile bending stress on the surface of the stem at cross-section X-X, assuming that there is only cement fixation below this cross-section (Figure 3). Assume that the joint contact force **J** and the abductor force **A** act at angles α and β, respectively, as shown. The (x, y) coordinates of the points *a* (the femoral head center), *b* (the assumed single attachment point of the abductors), and *c* (the center of the cross-section at X–X) are (a_x, a_y), (b_x, b_y), and (c_x, c_y), respectively.

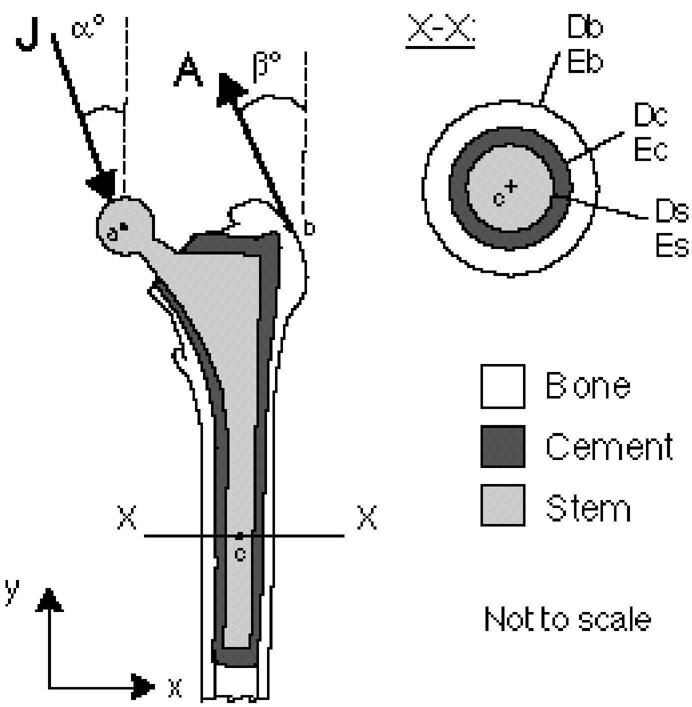

Figure 3: Two-dimensional model of a cemented hip prosthesis, showing the cross-section at the mid diaphysis (D = outside diameter; E = Young's modulus; subscripts *b*, *c*, and *s* refer to the bone, cement, and stem, respectively). Assume that the device is only fixed distally, below the section X-X.

B. **[5 points]** In terms of minimizing the risk of stem fracture for this situation, what would be the advantages of changing (i) the material; and (ii) the geometry of the stem?

4. (20 points total) Osteoporosis and Bone Fracture

Trabecular bone is known to become damaged after yielding. This damage is characterized by a reduction in both Young's modulus and yield strength. In a load sharing structure such as the vertebral body, damage of only some of the trabecular bone within the vertebral body can result in increased stresses in undamaged (intact) bone for subsequent loading and eventual bone fracture. One important issue is the effects of damage on modulus vs. strength reductions.

Consider an idealized vertebral body (Figure 4) under a uniaxial compressive load **P**, which is applied through a loading platens that remains perfectly horizontal.

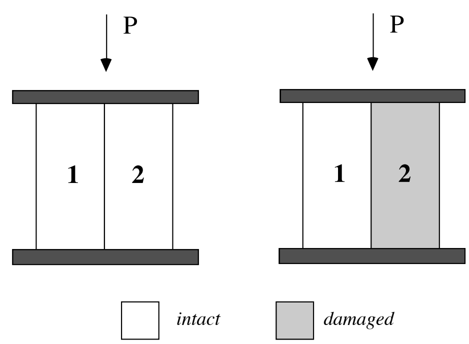

Figure 4: Schematic of vertebral body, without (left) and with (right) damage. $A_2 = A_1 = A/2$. $E_2 = E_1$ for fully intact; $E_2 = 0.08E_1$ for partially damaged; P = applied uniaxial compressive force. Top and bottom plates (darkly shaded) remain horizontal and parallel throughout the loading.

A. [7 points] If the bone in Region 1 remains intact with a modulus of $E_1 = E$, but the bone in Region 2 is damaged with a reduced modulus $E_2 = 0.08$ E, then calculate the stresses in each region after damage ($\sigma_{damage-1}$, $\sigma_{damage-2}$) in terms of the original (no damage) stresses (σ).
Hint: this structure behaves like elastic springs loaded in parallel.

B. **[7 points]** If the strength of the damaged bone is $\sigma_{y\text{-damage}} = 0.70\ \sigma_y$ (*i.e.* 70% of the original strength σ_y), then calculate the risk-factor *RF* (functional stress/strength) in each region before (RF_{intact}) and after ($RF_{damage\text{-}1}$; $RF_{damage\text{-}2}$) partial damage occurs.
Repeat this analysis assuming that the strength reduction with damage is equal to the modulus reduction (*i.e.* $\sigma_{y\text{-damage}} = 0.08\ \sigma_y$ after damage). Fill out the following Table with your results.

Risk Factor (RF)	**Region 1**	**Region 2**
RF intact		
RF damage ($\sigma_{y\text{-damage}} = 0.70\ \sigma_y$)		
RF damage ($\sigma_{y\text{-damage}} = 0.08\ \sigma_y$)		

C. **[6 points]** Give one clinical implication of these analyses.

5. (40 points total) Miscellaneous

A. [10 points] Fill in the following table of material properties with approximate values:

Material	*Young's modulus (MPa)*	*Compressive yield strength (MPa)*	*Tensile yield strength (MPa)*
Cortical bone †			
Tendon		—	—
UHMWPE		—	
PMMA			
Ti-6Al-4V alloy		—	

† for cortical bone, give values for longitudinal loading (for human bone)

B. [5 points] On a single graph (no specific values required), show how the maximum compressive contact stresses in UHMWPE depend on the polyethylene thickness for both an acetabular hip component and a tibial knee component. On the same graph, indicate how these two curves would change if the modulus of the UHMWPE were increased (by carbon-fiber reinforcement, for example).

C. [5 points] In the context of beam-on-elastic-foundation theory, explain briefly the concept of "flexible" vs. "rigid" behaviors. For a single beam of modulus E, moment of inertia I, length L, resting on an elastic foundation of "foundation modulus" K, "rigid" behavior is promoted by increasing or decreasing which parameters?

D. **[5 points]** Explain the mechanism of hydrodynamic lubrication. How does it differ from elasto-hydrodynamic lubrication? What lubrication mechanisms are thought to occur in synovial joints?

E. **[5 points]** For the equilibrium behavior of articular cartilage, does the compressive modulus increase or decrease with an increase in :

(i) collagen content

(ii) proteoglycan content

(iii) water content

(iv) Does the tensile modulus increase or decrease with an increase in collagen content?

(v) Does the shear modulus increase or decrease with an increase in collagen content?

F. **[5 points]** Plot a force-velocity curve for muscle, superimposed with a power-velocity curve, all for muscle contraction.

G. **[5 points]** Name the primary types of bone cells and describe briefly a remodeling cycle.

Additional work space (indicate clearly which question):