Saturday, December 11, 8:00–11:00 AM, 1999.

Answer all questions for a maximum of 100 points. Please write all answers in the space provided. If you need more space, there is an additional space at the end. Indicate your answer as clearly as possible for each question. Write your name at the top of each page as indicated. *Read each question very carefully!*

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1. (10 points total) Skeletal Loading

Figure 1 depicts a schematic of the proximal femur with the joint contact *J* and abductor *A* forces shown, during the single legged stance phase of gait. Show, with your own free-body diagram analysis, why the angle □ must always be greater than angle □.

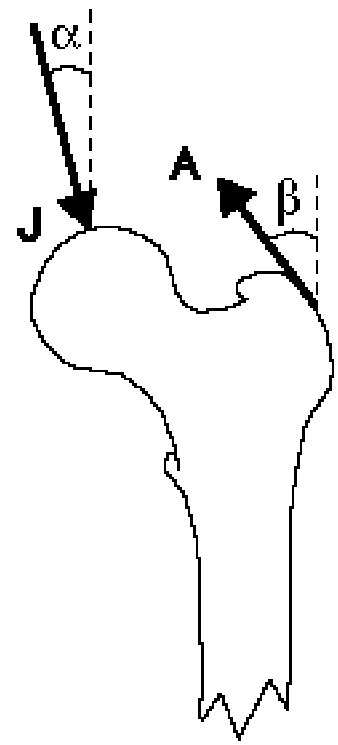


Figure 1

2. (20 points total) Dynamic Analysis of the Skeleton

Impact to the head during automobile accidents represents a serious injury. These injuries are particularly sensitive to the type of safety restraint used. Consider the special case where a seatbelt is used but without shoulder strap. A crash test dummy model is used to investigate head forces at impact. Assume that the upper torso and head act as a rigid body consisting of a uniform slender rod of mass *m* and length *L* (Figure 2). Immediately before vehicular impact ($t<T_o$), assume that the mass center of the torso/head complex is moving at a horizontal velocity *v*, and that at some time after impact ($t>T_1$), the head hits the dashboard when the torso is at an angle □ to the horizontal. During the period of vehicular impact ($T_o<t<T_1$), the vertical and horizontal forces at the hip were gathered as functions of time by strain gauges in the seatbelt (Figure 3). For this situation:

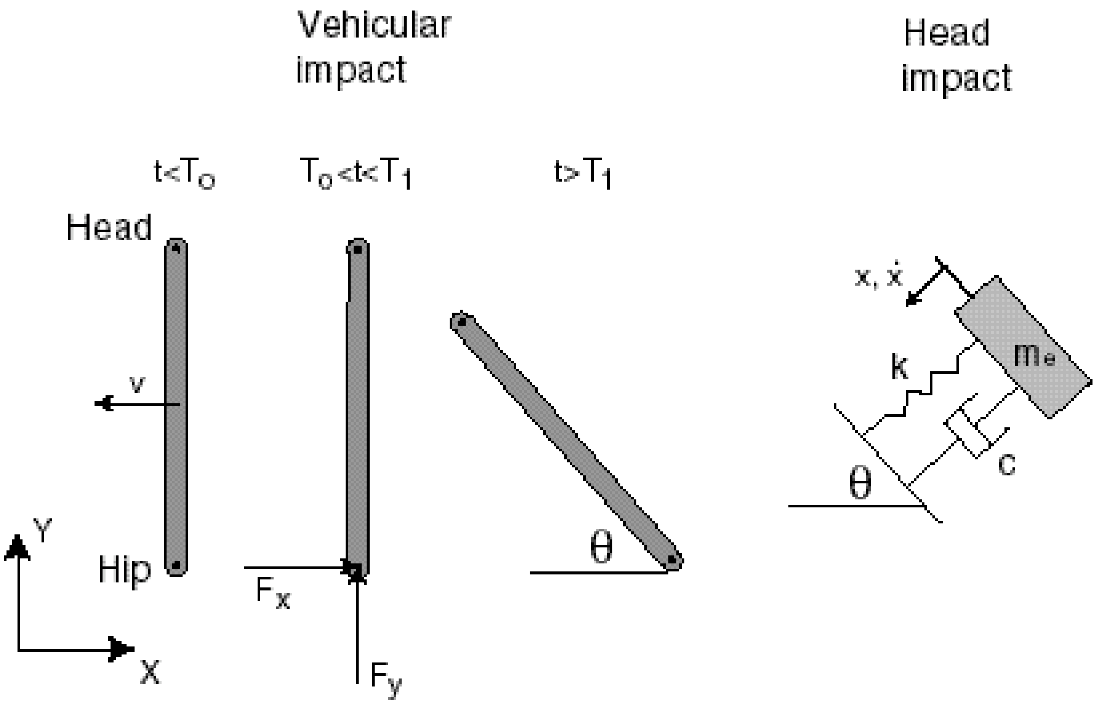


Figure 2

(i) [8 points] Calculate the horizontal and vertical velocities of the center of mass, v_x and v_y, respectively, as well as the angular velocity, ω, of the torso/head complex at T_1. Express these quantities in terms of *v* (the original horizontal velocity), F_x, F_y, T_o, T_1, m, and any geometrical parameters. Assume that the only appreciable forces acting on the head/torso complex are those measured at the hip, *i.e.* neglect gravity and any muscle forces, etc.

(ii) [5 points] Calculate the tangential velocity of the head just before head impact. Express in terms of v_x, v_y, ω, and θ.

NOTE: $\boldsymbol{v}_{head} = \boldsymbol{v}_{cm} + \boldsymbol{\omega} \times \boldsymbol{r}_{head/cm}$

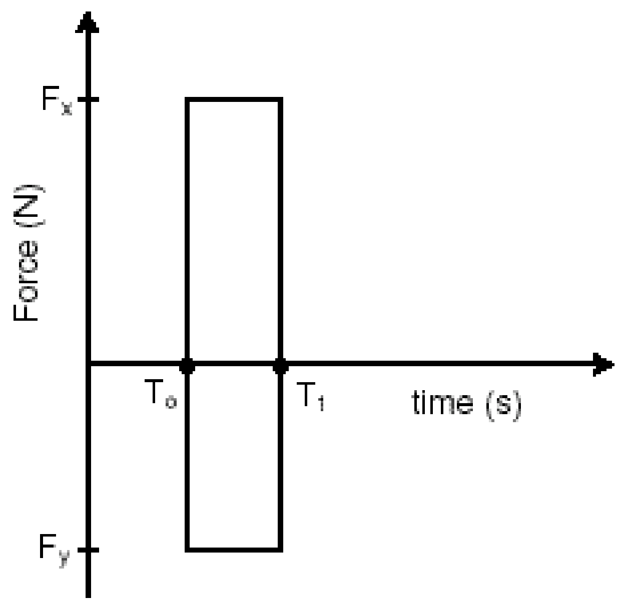


Figure 3

(iii) [7 points] If the head impact can be modeled using the spring-dashpot system as shown (Figure 2) where m_e is the effective mass of the impacting body, write out the equation of motion for this dynamic spring-dashpot impact model in terms of the parameters shown and the original horizontal velocity *v* of the head/torso complex. Assume this is a one-dimensional problem along the axis of the spring and dashpot.

3. (25 points total) Shear Lag Theory, Load Sharing, and Load Transfer

(i) [10 points] For shear-lag theory applied to two concentric cylinders loaded axially:

(a) draw the free-body diagrams of the relevant differential elements of the structure

(b) write out the equations for the kinematic assumptions

(c) write out the equations for the constitutive assumptions

(d) write out the boundary conditions

(ii) [5 points] From these equations, derive the overall governing differential equation:

$$\frac{d^2\square}{dx^2} - \left[\frac{\square\,(1+\square)\,\bar{D}\,G}{t\,E_1\,A_1}\right]\square = 0$$

where $\square$ is the shear strain in the interface material, x is the distance along the interface, $\square$ is the ratio of the axial stiffnesses (E_1A_1/E_2A_2), $\bar{D}$ is the average diameter of the interface, G is the shear modulus of the interface, and the subscripts refer to the axially loaded cylinders (1 is the outer cylinder, 2 is the inner cylinder).

(iii) [6 points] Sketch a typical graph of shear stress along the interface, assuming "flexible" behavior of the structure. Identify the load sharing and load transfer regions.

(iv) [4 points] What is the condition that optimizes the strength of the interface?

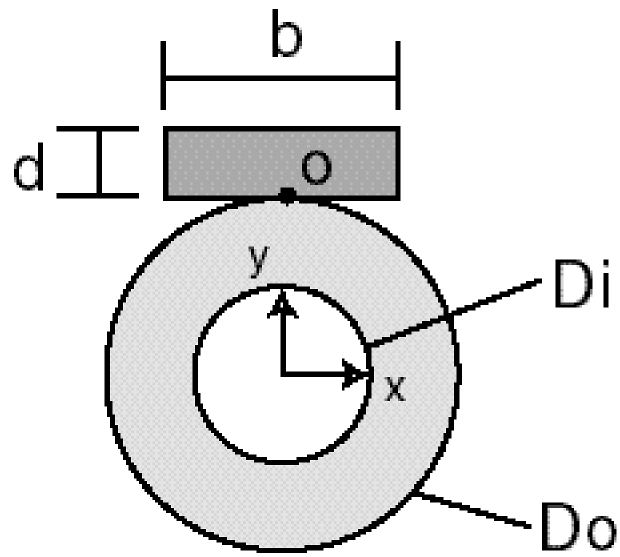
4. (25 points total) Composite Beam Theory and Implant Design

Use of bone plates vs. intramedullary (IM) rods comes with various trade-offs with respect to biomechanical performance. To appreciate this, work the following problem. Assume in all cases that the bone is fully healed, the bone and implant share loads according to composite beam theory, and the system is subjected to a pure bending moment *M* with tension on the top (plate) side.

D_o=30 mm
D_i=15 mm
d=7 mm
b=20 mm

(i) [5 points] Derive an expression for the location of the neutral axis for the bone-plate system shown here in cross-section (Figure 4). E_b — bone modulus; E_i — implant modulus.

(ii) [10 points] For point *o* on the bone periosteal surface, derive an expression for the bone stress at this point in the presence of the plate. Calculate the ratio of this stress to the stress at point *o* for the case where there is no plate.



(iii) [5 points] Next, express the stress at point *o* in the bone for the case of a solid IM rod that completely fills the intramedullary cavity. Calculate the ratio of this stress to the stress at point *o* for the case where there is no rod.

(iv) [5 points] According to these results, which system would you use (plate or rod), and why?

Figure 4

5. (20 points total) Miscellaneous

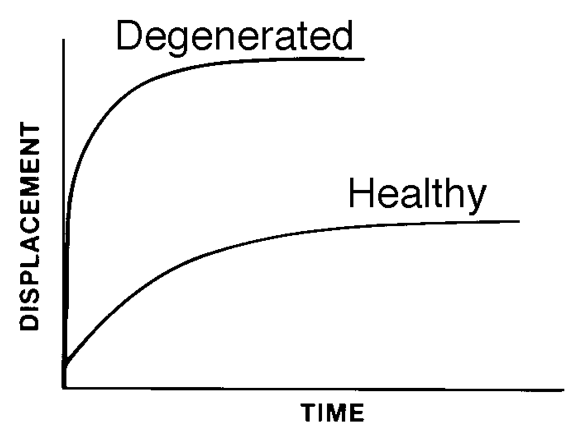
(i) [5 points]

Fill in the following table of *tensile* fatigue properties (in units of MPa) with approximate values:

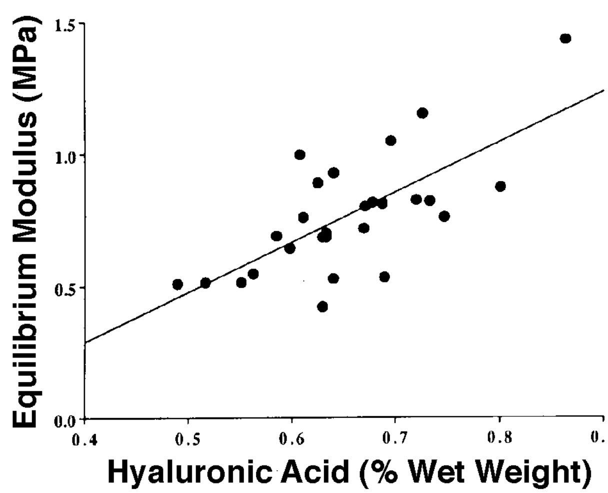
Material	*Fatigue Strength (MPa)*
Cortical bone †	
UHMWPE	
PMMA	
Ti-6Al-4V alloy	
CoCr alloy (forged)	
316L stainless steel	

† for cortical bone, give a typical mean value for human bone for longitudinal loading.

(ii) [5 points] The Figure below shows a comparison of the creep response to the same static load for a healthy vs. severely degenerated intervertebral disk. Give two interpretations of this graph.



(iii) [5 points] The Figure below shows the dependence of equilibrium compressive aggregate modulus for human patellar cartilage on hyaluronic acid content. State the main constituents of articular cartilage and explain the mechanisms underlying the behavior shown here.



(iv) [5 points] What are the four main types of bone cells and their main (or presumed) functions?

Additional work space (indicate clearly which question; use back side if necessary):