Operations Research II IEOR 161
University of California, Berkeley
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Solutions for Midterm 1

1. Let X_i be the indicator random variable taking value 1 if the i^{th} contestant pulls out one or two red tickets, 0 otherwise. Also, let X denote the number of prizes given out. We have $X = \sum_{i=1}^{30} X_i$ and thus

$$E[X] = \sum_{i=1}^{30} E[X_i] = \sum_{i=1}^{30} P\{X_i = 1\} = 30P\{X_1 = 1\}$$

where the last equation comes from the fact that each contestant has the same probability of getting at least one red ticket.

Now

$$P\{X_1 = 1\} = 1 - P\{X_1 = 0\} = 1 - P\{\text{contestant 1 pulls out 2 black tickets}\} = 1 - \frac{80 \times 79}{100 \times 99}$$

since there is no replacement. Therefore,

$$E[X] = 30 \times \left(1 - \frac{80 \times 79}{100 \times 99}\right)$$

2. Let X denote the number of heads that occur, F denote the event that the fair coin is chosen, and B the event that the biased coin is chosen.

   (a) Conditioning on which coin is chosen gives

$$\begin{aligned} E[X] &= E[X|F]P\{F\} + E[X|B]P\{B\} \\ &= (5 \times 0.5)(0.5) + (5 \times 0.7)(0.5) \\ &= 3 \end{aligned}$$

   where the second equation comes from the fact that $X|F$ is a binomial random variable with parameters $(5, 0.5)$, and $X|B$ is a binomial random variable with parameters $(5, 0.7)$.

   (b) We want to find $P\{F|X = 4\}$. Using Bayes rule gives

$$\begin{aligned} P\{F|X = 4\} &= \frac{P\{X = 4|F\}P\{F\}}{P\{X = 4|F\}P\{F\} + P\{X = 4|B\}P\{B\}} \\ &= \frac{\binom{5}{4}(0.5)^4(0.5) \times (0.5)}{\binom{5}{4}(0.5)^4(0.5) \times (0.5) + \binom{5}{4}(0.7)^4(0.3) \times (0.5)} \end{aligned}$$

3. Let X denote the number of flips Jack makes, Y the number of flips Jill makes. We have X and Y are geometric random variables with parameters 0.5 and p respectively. We want to find $P\{X > Y\}$.

   First Approach: Conditioning on the value of X we have

$$\begin{aligned} P\{X > Y\} &= \sum_{i=1}^{\infty} P\{X > Y | X = i\} P\{X = i\} \\ &= \sum_{i=2}^{\infty} P\{Y < i\}(1-0.5)^{i-1}(0.5) \\ &\qquad \text{since } X \text{ and } Y \text{ are independent, } P\{Y < 1\} = 0 \\ &= \sum_{i=2}^{\infty} (1 - P\{Y \geq i\})(0.5)^i \\ &= \sum_{i=2}^{\infty} (1 - (1-p)^{i-1})(0.5)^i \end{aligned}$$

   since $P\{Y \geq i\} = P\{$ first $i-1$ flips are tails $\} = (1-p)^{i-1}$.

   Second Approach: Conditioning on the value of Y we have

$$\begin{aligned} P\{X > Y\} &= \sum_{i=1}^{\infty} P\{X > Y | Y = i\} P\{Y = i\} \\ &= \sum_{i=1}^{\infty} P\{X > i\}(1-p)^{i-1} p \quad \text{since } X \text{ and } Y \text{ are independent} \\ &= \sum_{i=1}^{\infty} (0.5)^i (1-p)^{i-1} p \end{aligned}$$

   since $P\{X > i\} = P\{$ first i flips are tails $\} = (0.5)^i$.

   Third Approach: Conditioning on the outcome of the first flip of Jack, say Ja_1, and the outcome of the first flip of Jill, say Ji_1 gives

$$\begin{aligned} P\{X > Y\} &= P\{X > Y | Ja_1 = H, Ji_1 = H\}(0.5p) \\ &\quad + P\{X > Y | Ja_1 = H, Ji_1 = T\}(0.5(1-p)) \\ &\quad + P\{X > Y | Ja_1 = T, Ji_1 = H\}(0.5p) \\ &\quad + P\{X > Y | Ja_1 = T, Ji_1 = T\}(0.5(1-p)) \\ &= 0 + 0 + 1 \times (0.5p) + P\{X > Y\}(0.5(1-p)) \end{aligned}$$

   The above equation implies $P\{X > Y\} = \frac{p}{1+p}$.

4. Let A denote the moment when the pipe-smoking man first discovers that one of his matchboxes is empty, and L denote the number of matches left in the box in his left-hand pocket at A, R the number of matches left in the box in his right-hand pocket at A. For each $k = 1, 2, \ldots, N$, we want to find $P\{L = k \text{ or } R = k\}$.

We have

$$P\{L = k \text{ or } R = k\} = P\{L = k\} + P\{R = k\} = 2P\{L = k\}$$

since $\{L = k\} \bigcap \{R = k\} = \emptyset$ and $P\{L = k\} = P\{R = k\}$ by symmetry.
To find $P\{L = k\}$, we condition on from which matchboxes the last match is taken

$$\begin{aligned} P\{L = k\} &= P\{L = k | \text{ last match is from right-hand pocket}\}(0.5) \\ &\quad + P\{R = k | \text{ last match is from left-hand pocket}\}(0.5) \\ &= P\{\text{exactly } N - k \text{ of the first } N - 1 + N - k \text{ matches} \\ &\qquad \text{are taken from the left-hand pocket}\}(0.5) + 0 \\ &= \binom{2N - k - 1}{N - k}(0.5)^{N-k}(0.5)^{N-1} \times (0.5) \\ &= \binom{2N - k - 1}{N - k}(0.5)^{2N-k} \end{aligned}$$

Therefore the desired probability is

$$P\{L = k \text{ or } R = k\} = 2 \times \binom{2N - k - 1}{N - k}(0.5)^{2N-k}$$