IEOR 172 - Probability and Risk Analysis for Engineering
GSI Solutions for Midterm 1

Note: These solutions *may* have errors as these are my solutions and not Professor Shanthikumar's solutions. That said, I'm fairly confident that my answers are correct. -Shea

Q1:

$$\begin{aligned} P(n\text{th coupon collected is new}) &= \sum_{i=1}^{m} P(n\text{th new}|n\text{th is type } i)P(n\text{th is type } i) \\ &= \sum_{i=1}^{m}(1-p_i)^{n-1}p_i \end{aligned}$$

Q2, Part 1: Assumption: Each trial is independent. This assumption lets us define X, the random variable that tells us the number of trials to the first success, as a geometric random variable with success probability p. Thus let $X \sim \text{Geom}(p)$. Then by definition,

$$\mathbb{E}[X] = \frac{1}{p}$$

but note that we are given $\mathbb{E}[X] = 5$ which means $p = \frac{1}{5}$. Using this fact, we can calculate the variance:

$$\text{Var}(X) = \frac{1-p}{p^2} = \frac{\frac{4}{5}}{\frac{1}{25}} = 20$$

After knowing the variance, we can calculate the second moment as follows:

$$\begin{aligned} \text{Var}(X) &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ \mathbb{E}[X^2] &= \text{Var}(X) + (\mathbb{E}[X])^2 \\ &= 20 + (5)^2 \\ &= 45 \end{aligned}$$

Now define Y as the random variable that tells us the number of failed trials before the first success. Note that $Y = X - 1$. This means the first moment, or the expected value, is

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[X-1] \\ &= \mathbb{E}[X] - 1 \\ &= 5 - 1 \\ &= 4. \end{aligned}$$

The variance is

$$\begin{aligned} \text{Var}(Y) &= \text{Var}(X-1) \\ &= \text{Var}(X) \\ &= 20 \end{aligned}$$

which means the second moment is

$$\begin{aligned}\mathbb{E}[Y^2] &= \text{Var}(Y) + (\mathbb{E}[Y])^2 \\ &= 20 + 4^2 \\ &= 36.\end{aligned}$$

Notice we can also calculate the second moment directly as follows

$$\begin{aligned}\mathbb{E}[Y^2] &= \mathbb{E}[(X-1)^2] \\ &= \mathbb{E}[X^2 - 2X + 1] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X] + 1 \\ &= 45 - 2 \cdot 5 + 1 \\ &= 36.\end{aligned}$$

Q2, Part 2: Note that to find the probability mass function of X (it is similar for Y), we need to find

$$\begin{aligned}P(X = x) &= \sum_y P(X = x|Y = y)P(Y = y) \\ &= \sum_{y:P(Y=y)>0} \frac{P(X = x \cap Y = y)}{P(Y = y)} P(Y = y) \\ &= \sum_{y:P(Y=y)>0} P(X = x \cap Y = y)\end{aligned}$$

which means to find the pmf for $X = x$, we just need to add all the $f_{X,Y}$ such that $X = x$. Thus, for example,

$$P(X = 1) = f_{X,Y}(1,1) + f_{X,Y}(1,2) + f_{X,Y}(1,3) + f_{X,Y}(1,4) = .06 + .09 + .12 + .03 = .3$$

Doing the math gives us the following table

$P(X = 1) = .3$	$P(Y = 1) = .2$
$P(X = 2) = .4$	$P(Y = 2) = .3$
$P(X = 3) = .06$	$P(Y = 3) = .4$
$P(X = 4) = .09$	$P(Y = 4) = .1$
$P(X = 5) = .12$	
$P(X = 6) = .03$	

These two random variables are not statistically independent. Notice that the probability of $X = 5$ depends on the value of Y. Because $P(X = 5|Y = 3) = 0.12$ and $P(X = 5|Y \neq 3) = 0$, X depends on Y and thus they are not independent.

Now $Z = X + Y$. Thus the mean or the expected value is

$$\begin{aligned}\mathbb{E}[Z] &= \sum_{y=1}^{4}\sum_{x=1}^{6}(x + y) \cdot f_{X,Y}(x, y) \\ &= (1 + 1)f_{X,Y}(1,1) + (1 + 2)f_{X,Y}(1,2) + \cdots + (6 + 4)f_{X,Y}(6,4) \\ &= 4.82\end{aligned}$$

the second moment is

$$\begin{aligned}\mathbb{E}[Z^2] &= \sum_{y=1}^{4}\sum_{x=1}^{6}(x+y)^2 \cdot f_{X,Y}(x,y) \\ &= (1+1)^2 f_{X,Y}(1,1) + (1+2)^2 f_{X,Y}(1,2) + \cdots + (6+4)^2 f_{X,Y}(6,4) \\ &= 26.68\end{aligned}$$

and the variance is simply

$$\begin{aligned}\mathrm{Var}(Z) &= \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2 \\ &= 26.68 - (4.82)^2 \\ &= 3.4476\end{aligned}$$