Operations Research II IEOR 161
University of California, Berkeley
Spring 2004
Solutions for Midterm 2

1. (a) Let T_j denote the amount of time when exactly j components are working. Since everything starts over when a failure occurs, T_j also denotes the time until the next failure given that we start with j new components, and therefore T_j is exponentially distributed with rate $j \times \lambda$ (i.e. $\text{dist}(T_j) = \text{dist}(\min(X_1, \ldots, X_j))$ where $\{X_j\}$ are independent exponential with rate λ.)
Thus,
$$E[\text{profit}] = E[10(T_4 + T_3) + 5T_2 + 2T_1] = 10\left(\frac{1}{4\lambda} + \frac{1}{3\lambda}\right) + \frac{5}{2\lambda} + \frac{2}{\lambda}$$

   (b) Let L_i denote the lifetime of component i, then
$$P\{\text{no component fails in the first hour}\} = P\{\min(L_1, L_2, L_3, L_4) > 1\} = e^{-4\lambda \cdot 1}$$
since $\min(L_1, L_2, L_3, L_4)$ is exponentially distributed with rate 4λ and recall that if Z is exponentially distributed with rate α (i.e. with mean $1/\alpha$), and c is a positive number, then $P\{Z > a\} = e^{-\alpha a}$.

2. (a)
$$P\{\text{a customer will buy a house}\} = P\{X_j > y\} = e^{-\frac{1}{\mu}y}$$
where X_j is exponentially distributed with mean μ (i.e. with rate $1/\mu$.)

   (b) Since each arrival is independently willing to buy a house with probability $e^{-\frac{1}{\mu}y}$ (from (a)), we have the process $\{N_B(t), t \geq 0\}$ is a Poisson process with rate $\lambda e^{-\frac{1}{\mu}y}$ where $N_B(t)$ denotes the number of arrivals who are willing to buy houses by time t. Therefore, the distribution of the inter-arrival time of this process is exponential with rate $\lambda e^{-\frac{1}{\mu}y}$.

   (c) Consider first that if we sell 100 houses, each at price y, then the total revenue will be $100\,y$. Now let S_n^B denote the time of the n^{th} arrival of process $\{N_B(t), t \geq 0\}$. We have S_n^B is the sum of n independent exponential random variables, each of which has mean $1/(\lambda e^{-\frac{1}{\mu}y})$ (i.e. S_n^B is gamma$(n, \lambda e^{-\frac{1}{\mu}y})$) and hence
$$E[S_n^B] = \frac{n}{\lambda e^{-\frac{1}{\mu}y}}$$
Since the cost is c per day, the total cost until we sell all 100 houses is $c \times S_{100}^B$. The expected total profit is therefore
$$E[\text{revenue}] - E[\text{cost}] = E[100y] - E[c \times S_{100}^B] = 100y - \frac{100c}{\lambda e^{-\frac{1}{\mu}y}} \tag{1}$$

(d) Taking the first derivative of (1) with respect to y and setting it equal to 0 gives

$$100 - \frac{100c}{\lambda\mu} e^{\frac{1}{\mu}y} = 0$$

Solving the above equation yields

$$y^* = \mu \ln\left(\frac{\lambda\mu}{c}\right)$$

Note that the second derivative of (1) with respect to y is

$$-\frac{100c}{\lambda\mu^2} e^{\frac{1}{\mu}y} < 0$$

and hence the above y^* is the unique optimal price. Now the optimal expected profit is obtained by substituting y^* into (1), i.e. the optimal expected profit is

$$100\mu \ln\left(\frac{\lambda\mu}{c}\right) - 100\mu$$

3. Let $N^R(t)$ and $N^G(t)$ denote the number of arrivals by time t whose t-shirts are red and green, respectively. Also, let $\{T_i^R\}$ and $\{T_i^G\}$ denote the inter-arrival times of processes $\{N^R(t), t \geq 0\}$ and $\{N^G(t), t \geq 0\}$ respectively.

   Since each time we have an arrival, everything starts over (each process is a Poisson process), we have the distribution of the inter-arrival times of process $\{N^R(t) + N^G(t), t \geq 0\}$ is the same as the distribution of $\min(T_i^R, T_j^G)$ which is exponential with rate $\lambda + \mu$.

4. See solutions for homework 11.

5. Let $N(t)$ denote the number of arrivals by time t of the Poisson process with rate $\lambda = 2$.

   (a) Note that $N(t+s) - N(t)$ is Poisson with mean λs.

$$\begin{aligned} P\{N(1) = 0, N(10) - N(9) = 0\} &= P\{N(1) = 0\} \times P\{N(10) - N(9) = 0\} \\ &\qquad \text{by independent increments} \\ &= \frac{e^{-\lambda \cdot 1}(\lambda \cdot 1)^0}{0!} \times \frac{e^{-\lambda \cdot (10-9)}(\lambda \cdot (10-9))^0}{0!} \\ &= e^{-2\lambda} = e^{-4} \end{aligned}$$

   (b) Recall that given $N(t) = n$, the arrival time of each of the first n arrivals is independent uniform$(0, t)$. So given $N(10) = 5$, each of the first 5 arrivals independently arrives uniformly over (0,10) (hence for each of these 5 arrivals, the probability that this arrival arrives during the first hour is 1/10, during the last hour is 1/10, and after the first hour but before the last hour is 8/10.) Thus,

$$\begin{aligned}
&P\{N(1)=1, N(10)-N(9)=1|N(10)=5\}\\
&\qquad = P\{N(1)=1, N(10)-N(9)=1, N(9)-N(1)=3|N(10)=5\}\\
&\qquad = \frac{5!}{1!3!1!}\times\frac{1}{10}\times\left(\frac{8}{10}\right)^3\times\frac{1}{10}
\end{aligned}$$

where the last line comes from multinomial distribution.

(c) Recall that for any two events A and B, we have $(A\cap B)^C = A^C\cup B^C$, $P\{A\} = 1-P\{A^C\}$, and $P\{A\cup B\} = P\{A\}+P\{B\}-P\{A\cap B\}$.
Now

$$\begin{aligned}
&P\{N(1)\geq 1, N(10)-N(9)\geq 1|N(10)=5\}\\
&\qquad = 1-P\{N(1)=0\cup N(10)-N(9)=0|N(10)=5\}\\
&\qquad = 1-\Big(P\{N(1)=0|N(10)=5\}+P\{N(10)-N(9)=0|N(10)=5\}\\
&\qquad\qquad -P\{N(1)=0, N(10)-N(9)=0|N(10)=5\}\Big)\\
&\qquad = 1-\left(\left(\frac{9}{10}\right)^5+\left(\frac{9}{10}\right)^5-\left(\frac{8}{10}\right)^5\right) \qquad (2)
\end{aligned}$$

Alternatively, let the first 5 arrivals be A, B, C, D and E, and let S_k denote the arrival time of k for $k\in\{A,B,C,D,E\}$, $S_{\min}=\min(S_A,\ldots,S_E), S_{\max}=\max(S_A,\ldots,S_E)$. Now

$$\begin{aligned}
&P\{N(1)\geq 1, N(10)-N(9)\geq 1|N(10)=5\}\\
&\quad = \sum_{j\neq k\in\{A,\ldots,E\}} P\{N(1)\geq 1, N(10)-N(9)\geq 1, S_j=S_{\min}, S_k=S_{\max}|N(10)=5\}\\
&\quad = 5\cdot 4\cdot P\{N(1)\geq 1, N(10)-N(9)\geq 1, S_A=S_{\min}, S_B=S_{\max}|N(10)=5\}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{by symmetry}\\
&\quad = 20\int_0^1 P\{N(1)\geq 1, N(10)-N(9)\geq 1, S_A=S_{\min}, S_B=S_{\max}|N(10)=5, S_A=x\}\frac{1}{10}dx\\
&\quad = 20\int_0^1\int_9^{10} P\{N(1)\geq 1, N(10)-N(9)\geq 1, S_A=S_{\min}, S_B=S_{\max}\\
&\qquad\qquad\qquad |N(10)=5, S_A=x, S_B=y\}\frac{1}{10}\cdot\frac{1}{10}dydx\\
&\quad = 20\int_0^1\int_9^{10} P\{x\leq S_C, S_D, S_E\leq y|N(10)=5\}\frac{1}{10}\cdot\frac{1}{10}dydx\\
&\quad = 20\int_0^1\int_9^{10}\left(\frac{y-x}{10}\right)^3\frac{1}{10}\cdot\frac{1}{10}dydx
\end{aligned}$$

Integrating the last equation gives the same result as in (2).