Operations Research II, IEOR161

University of California, Berkeley

Midterm Exam I Suggested Solutions, 2009

1. (a) Poisson arrivals take exponential amount of time. Thus, arrival of the first man is an exponential with rate $\lambda = 2$ and the arrival of the first woman is exponential with rate $\mu = 3$. Therefore, the probability the first arrival is a man is $\frac{\lambda}{\lambda+\mu} = \frac{2}{2+3} = \frac{2}{5}$.

   (b) Exponential random variables are memoryless, so the probability of a male arriving before a female at any point in the process will be $\frac{2}{5}$. Therefore, the probability that the first three arrivals are male is $\left(\frac{2}{5}\right)^3$.

   (c) Let X be the amount earned from the first customer and let $Y = 0$ if the first customer is male and $Y = 1$ if the first customer is female. Then,

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X|Y=0]P(Y=0) + \mathbb{E}[X|Y=1]P(Y=0) \\
&= 10 \cdot \frac{2}{5} + 20 \cdot \frac{3}{5} \\
&= 16
\end{aligned}$$

2. (a) A mean time of 0.75 means the rate of the exponential is $\frac{1}{0.75} = \frac{4}{3}$. Let the completion time of student i be denoted by T_i where $T_i \sim \exp(\frac{4}{3}),\ i = 1, \ldots, 80$. The probability the ith student finishes early is

$$P(T_i < 1) = 1 - e^{-\frac{4}{3}}.$$

   Let $X_i = 1$ if the ith student leaves early and $X_i = 0$ otherwise. Note that $P(X_i = 1) = 1 - e^{-\frac{4}{3}}$.

   Let $X = \sum_{i=1}^{80} X_i$. Note that X is then the number of students that leave early. Furthermore, we can think of X is a binomial random variable with 80 trials and a probability $1 - e^{-\frac{4}{3}}$ of success. With this set up, we are thus looking for the probability $X = 20$, which is simply

$$P(X = 20) = \binom{80}{20} \left(1 - e^{-\frac{4}{3}}\right)^{20} \left(e^{-\frac{4}{3}}\right)^{60}$$

   (b) The number of students that finish early is the random variable X we defined in part (a). Therefore, the expected number of students that finish early is simply $\mathbb{E}[X] = np$, the expected value of a binomial random variable. From part (a), $n = 80$ and $p = 1 - e^{-\frac{4}{3}}$, therefore $\mathbb{E}[X] = 80\left(1 - e^{-\frac{4}{3}}\right)$.

3. (a) There are two ways to approach this problem. First, the hard way, which is conditioning on the departure time of the first service. The time of the first service, let it be T, is the minimum of the service times. Therefore, $T \sim \exp(\mu_1 + \mu_2)$. Given we know the time of the departure, $T = t$, then the probability two arrivals come in the time $(0, t)$ is poisson distributed with mean λt. In other words,

$$P(\text{two arrivals in } (0,t) | T = t) = e^{-\lambda t} \frac{(\lambda t)^2}{2!}$$

Therefore, using conditional probability,

$$\begin{aligned} P(\text{two arrivals}) &= \int_0^\infty P(\text{two arrivals in } (0,t)|T = t) f_T(t)\, dt \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^2}{2!} (\mu_1 + \mu_2) e^{-(\mu_1+\mu_2)t}\, dt \end{aligned}$$

The easy way is to remember that we are dealing with exponential service and arrival times, which are memoryless and independent. The probability of an arrival occurring before a service is $\frac{\lambda}{\lambda+\mu_1+\mu_2}$. Thus the probability of a service occurring before an arrival is the complement, or $\frac{\mu_1+\mu_2}{\lambda+\mu_1+\mu_2}$. Therefore,

$$\begin{aligned} & P(2 \text{ arrivals at first departure}) \\ = \; & P(\text{1st arrival before 1st departure}) \times P(\text{2nd arrival before 1st departure}) \\ & \times P(\text{1st departure before 3rd arrival}) \\ = \; & P(\text{arrival before departure})^2 \times P(\text{departure before arrival}) \\ = \; & \left(\frac{\lambda}{\lambda + \mu_1 + \mu_2}\right)^2 \left(\frac{\mu_1 + \mu_2}{\lambda + \mu_1 + \mu_2}\right) \end{aligned}$$

Solving the integral in the hard way will give the same answer.

(b) Let T_i denote the time it takes for both servers to be busy given that there are currently i idle servers. We are looking for $\mathbb{E}[T_2]$.

$$\begin{aligned} \mathbb{E}[T_2] &= \frac{1}{\lambda} + \mathbb{E}[T_1] \\ \mathbb{E}[T_1] &= \frac{1}{2}\mathbb{E}[T_1|\text{1st Arrival entered server 1}] + \frac{1}{2}\mathbb{E}[T_1|\text{Server 2}] \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{E}[T_1|\text{1st Arrival entered server 1}] &= \mathbb{E}[\text{Time to 1st event}] + \mathbb{E}[\text{Remaining time}] \\ &= \frac{1}{\lambda + \mu_1} + 0 \times \frac{\lambda}{\lambda + \mu_1} + \mathbb{E}[T_2] \times \frac{\mu_1}{\lambda + \mu_1} \end{aligned}$$

We get a similar result if the first arrival enters server 2. Putting it all together gives us

$$\mathbb{E}[T_2] = \frac{\frac{1}{\lambda} + \frac{1}{2}\left(\frac{1}{\lambda+\mu_1} + \frac{1}{\lambda+\mu_2}\right)}{1 - \frac{1}{2}\left(\frac{\mu_1}{\lambda+\mu_1} + \frac{\mu_2}{\lambda+\mu_2}\right)}$$