IEOR 162
Linear Programming
Spring 2007

Midterm Solutions

1.

a. False. Consider the LP $\max\{x_1 + x_2 : x_1 + x_2 \leq 2, x_1, x_2 \geq 0\}$, for which one optimal solution is $x_1 = x_2 = 1$, which is clearly not a basic feasible solution (since only 1 of the 3 constraints is active at that point).

b. False. The problem could be unbounded, in which case x is not an optimal solution.

c. False. The correct statement is x is a basic *feasible* solution if and only if x is an extreme point.

d. False. Consider the LP $\max\{x_1 + x_2 : x_1 + x_2 \leq 2, x_1, x_2 \geq 0\}$, for which the following two points are feasible: $x^1 = (2, 0)$ and $x^2 = (0, 2)$. Now let $\alpha = 2$, then $\alpha x^1 + (1 - \alpha)x^2 = (4, 0) - (0, 2) = (4, -2)$, which is clearly not a feasible solution.

e. False. At each iteration of the simplex method, it is possible for all of the basic variables' values to change as well as one non-basic variable.

2.

a. Standard Form:

$$\begin{array}{rlrrrrrrrl}
-\max & -5x_1 & & & & & & & & \\
\text{s.t.} & 2x_1 & & - x_3^+ & + x_3^- & + s_1 & & & = & 0 \\
& & x_2 & + 3x_3^+ & - 3x_3^- & & - e_2 & & = & 0 \\
& x_1 & + x_2 & & & & & + s_3 & = & 0 \\
& 6x_1 & - 7x_2 & & & & & & = & 0 \\
& & & & & & & x_1, x_2, x_3^+, x_3^-, s_1, e_2, s_3 & \geq & 0
\end{array}$$

b. By setting the first three constraints of (P1), which are clearly linearly independent, to equality we get that a basic feasible solution is $(x_1, x_2, x_3) = (0, 0, 0)$.

c. $(0, 0, 0)$ is optimal because $x_1 \geq 0$ is a constraint and we are trying to minimize x_1.

3. a. To implement the simplex method using tableaus from a specified starting basis, we need to construct the initial tableau. Compute the reduced costs using the familiar formula $\bar{c}_N = c_N - c_B A_B^{-1} A_N$ and put the negative reduced costs in the top row of the tableau (and put 0 for all basic variables in the top row). In the constraints, simply put $A_B^{-1}A$ under the variables and $A_B^{-1}b$ for the right hand side. Finally, compute the objective using the formula $c_B A_B^{-1} b$. Using these computations, we get the initial tableau and can then do our two simplex iterations. Finally, note that I converted the problem to maximization before solving.

z	x_1	x_2	x_3	x_4 ↓	x_5	RHS	ratio
1	0	0	1.25	−.25	−7.375	−2	
0	1	0	.25	−.25	.875	1	−
0	0	1	.25	[.75]	.125	2	2.667

z	x_1	x_2	x_3	x_4	x_5 ↓	RHS	ratio
1	0	.333	1.333	0	−7.333	−1.333	
0	1	.333	.333	0	[.917]	1.667	1.818
0	0	1.333	.333	1	.167	2.667	16

z	x_1	x_2	x_3	x_4	x_5	RHS
1	8	3	4	0	0	12
0	1.091	.364	.364	0	1	1.818
0	.182	1.273	.273	1	0	2.364

b. The current basic solution after two iterations is $(0,0,0,\frac{26}{11},\frac{20}{11})$. This is an optimal solution because all of the reduced costs are non-positive. This leads to an optimal value of 12, which means the original problem has an optimal value of −12.

4. a.

Add the following data item to the list of data already described in the problem:
w_{ij} = 1 if a worker who starts on day i is working on day j, 0 otherwise

Variables:
x_i = number of workers to hire who start on day i
y_i = amount to produce on day i

Formulation:

$$
\begin{array}{llcll}
\max & \sum_{i=1}^{7}[(f-c)y_i - ex_i - p(d_i - y_i)] & & & \\
\text{s.t.} & y_i & \leq & d_i & i=1,\dots,7 \quad (1) \\
 & y_i & \leq & u\sum_{j=1}^{7} w_{ij}x_j & i=1,\dots,7 \quad (2) \\
 & y_i & \leq & um & i=1,\dots,7 \quad (3) \\
 & x_i, y_i & \geq & 0 & i=1,\dots,7 \quad (4)
\end{array}
$$

Since the problem states that we will hold no inventory, one can assume that we will never produce more than the demand amount on any given day. This also means that we will sell everything that we produce as long as we enforce the constraint that we can't produce more than demand. Thus, the objective maximizes the revenue received per product produced less the cost of production less the cost of labor less the penalty for unsatisfied demand. Constraint set 1 enforces the condition that we not produce more than demand. Constraint set 2 sets an upper bound on production based on the number of workers present on a given day. Constraint set 3 sets an absolute upper bound on production equal to the number of machines multiplied by the maximum production rate per machine. Finally constraint set 4 establishes that the variables are all non-negative.

It is possible to interpret one aspect of this problem differently. The problem states that there are m machines available, which means you could interpret the problem to be setting a bound on the number of workers that can be present each day (equal to m). If we include this set of constraints, we can leave out constraint set 3 in the previous formulation and replace them with the following constraints:

$$\sum_{j=1}^{7} w_{ij}x_j \quad \leq \quad m \quad i=1,\dots,7$$

b. Choose $x_i = y_i = 0, i = 1,\dots,7$, which corresponds to hiring no workers and producing nothing. Then the objective value is equal to $-p\sum_{i=1}^{7} d_i$.