Midterm Solutions

1.

a. *False.* Linear programs can be infeasible or unbounded. Also, a linear program can have an optimal solution, but no extreme point optimal solutions (i.e. $\max\{x_1 + x_2 \text{ s.t. } x_1 + x_2 = 1\}$).
b. *True.* Note that $x = 0$ is always a feasible solution, so this problem cannot be infeasible. Unbounded and having an optimal solution are the only other choices.
c. *False.* If the optimal basic feasible solution is degenerate then we could have a non-basic variable with 0 reduced cost, but not be able to move in the direction specified by increasing that non-basic variable.
d. *False.* The definition of d being a direction of unboundedness is that we have $d \geq 0, d \neq 0$, and $Ad = 0$. This says nothing about what cd is.
e. *False.* If a basic solution is degenerate, then a basic variable could be equal to 0 as well.

2.

a. There are multiple possibilities for this question but the answer that was easiest to determine is: $a > 0, b > 0, e \geq 0$.
b. $e = 0$.
c. $b < 0, c \leq 0, e \geq 0$.
d. $b = 0, c = 1, e \geq 0$.

3.

First, put the model into standard form with artificial variables (remember that since x_3 is unrestricted to replace it with $x_3^+ - x_3^-$):

$$\begin{aligned}
\max\ & 4x_1 + 2x_2 + 8x_3^+ - 8x_3^- \qquad - Ma_1 \\
\text{s.t. } & 2x_1 - x_2 + 3x_3^+ - 3x_3^- + x_4 = 30 \\
& x_1 + 2x_2 + 4x_3^+ - 4x_3^- + a_1 = 40 \\
& x_1, x_2, x_3^+, x_3^-, x_4, a_1 \geq 0
\end{aligned}$$

Tableau implementation of the big-M simplex method:

	z	x_1	x_2 ↓	x_3^+	x_3^-	x_4	a_1	RHS	ratio
Row 0	1	-4	-2	-8	8	0	M	0	
Row 0′	1	$-4-M$	$-2-2M$	$-8-4M$	$8+4M$	0	0	$-40M$	
	0	2	-1	3	-3	1	0	30	–
	0	1	$\boxed{2}$	4	-4	0	1	40	20

Note that x_2 was chosen to enter because it is the most convenient variable to choose. x_1 or x_3^+ could also have been chosen and you would eventually get to a feasible solution.

z	x_1	x_2	x_3^+	x_3^-	x_4	a_1	RHS
1	-3	0	-4	4	0	$M+1$	40
0	$\frac{5}{2}$	0	5	-5	1	$\frac{1}{2}$	50
0	$\frac{1}{2}$	1	2	-2	0	$\frac{1}{2}$	20

We have driven a_1 out of the basis, which implies that this tableau represents a feasible solution. The solution described is $x = (x_1, x_2, x_3^+, x_3^-, x_4) = (0, 20, 0, 0, 50)$.

4.
a.

It is not actually possible to model this problem via linear programming because of the piecewise linear cost function. In general, you can model a piecewise linear function using linear programming if you are trying to maximize a concave function or minimize a convex function. This problem you are trying to maximize a piecewise linear convex function, which cannot be done via linear programming. Points were not taken off for not correctly modeling the cost function. We looked to see if you could successfully model the other constraints and we gave you 5 extra points if you chose to model the problem in the way we had intended before we realized that you could not model the problem as an LP. Here is the model we had in mind:

Variables:

$$\begin{array}{rcl} t_i & = & \text{number of toy } i \text{ to produce} \\ x_j & = & \text{number of hours to run line } j \\ y & = & \text{number of extra labor hours used} \\ w_k & = & \text{amount of wood purchased at price point } k \end{array}$$

Formulation:

$$\begin{array}{llcll} \max & 5t_1 + 6t_2 - 10y - 4w_1 - 3w_2 - 2.5w_3 & & & \\ \text{s.t.} & w_1 + w_2 + w_3 & = & 4x_1 + 5x_2 & (1) \\ & t_1 & = & x_1 + 3x_2 & (2) \\ & t_2 & = & 2x_1 + x_2 & (3) \\ & x_1 + x_2 & \leq & 300 + y & (4) \\ & y & \leq & 50 & (5) \\ & w_1 & \leq & 800 & (6) \\ & w_2 & \leq & 200 & (7) \\ & \text{all variables} & \geq & 0 & \end{array}$$

The objective maximizes total profit, which is revenue minus costs. Constraint 1 gives the relationship between the total wood purchased and the total wood needed to run the two lines. Constraints 2 and 3 indicate how much we get of each toy by running the two lines. Constraint 4 ensures that we stay within the 300 regular labor hours or if we do go over that the penalty term y is increased. Constraints 5, 6, and 7 place appropriate bounds on y, w_1, and w_2 respectively. Finally, all of our variables must be non-negative.

b.
There are obviously many feasible solutions to this model. One possible solution is to set all variables equal to 0, which results in an objective value of 0.