IEOR 165 Midterm Solution Spring 2004

Q1.

$$f(x) = \begin{cases} e^{-(x-b)} & x \geq b \\ 0 & \text{otherwise} \end{cases}$$

(1)

$$\begin{aligned} E[X] &= \int_b^\infty x e^{-(x-b)} dx = b + 1 \\ b &= E[X] - 1 \\ B_{MOM} &= \overline{X} - 1 \end{aligned}$$

(2)

$$\begin{aligned} E[B_{MOM}] &= E[\overline{X} - 1] = E[\overline{X}] - 1 = E[X] - 1 = b \\ &\text{therefore } B_{MOM} \text{ is an unbiased estimator} \\ Var(B_{MOM}) &= Var(\overline{X} - 1) = Var(\overline{X}) = \frac{Var(X)}{n} \\ Var(X) &= E[X^2] - E^2[X] = \int_b^\infty x^2 e^{-(x-b)} dx - E^2[X] \\ &= b^2 + 2b + 2 - (b+1)^2 = 1 \\ Var(B_{MOM}) &= \frac{1}{n} \end{aligned}$$

(3)

$$\mathcal{L}(\underline{\mathcal{X}}; b) = \prod_{i=1}^{n} f(X_i; b) = \begin{cases} e^{nb} exp(-\sum x_i) & \text{if } b < min(X_1, X_2, \ldots, X_n) \\ 0 & otherwise \end{cases}$$

Thus, $\mathcal{L}(\underline{X}; b)$ is maximized when b is as large as possible but not exceeding $min(X_1, X_2, \ldots, X_n)$, that is when $b = min(X_i : i = 1, \ldots, n)$. Therefore, the maximum likelihood estimator is $B_{MLE} = min(X_i : i = 1, \ldots, n) = X_{[1]}$.

(4)

$$P(X > y) = \int_y^\infty e^{-x+b}dx = e^{-y+b}$$

$$F_{X_{[1]}}(y) = P(X_{[1]} \le y) = 1 - P(X_{[1]} > y) = 1 - \prod_{i=1}^{n} P(X_i > y) = 1 - (e^{-y+b})^n$$

$$f_{X_{[1]}}(y) = \frac{F_{X_{[1]}}(y)}{dy} = n(e^{-y+b})^n$$

$$E[B_{MLE}] = E[X_{[1]}] = \int_b^\infty yn(e^{-y+b})^n dy = b + \frac{1}{n} \ne b$$

therefore B_{MLE} is biased and $c = -\frac{1}{n}$.

$$Var(B_{MLE}) = E[X_{[1]}^2] - E^2[X_{[1]}] = \int_b^\infty y^2 n(e^{-y+b})^n dy - E^2[X_{[1]}]$$

$$= b^2 + \frac{2b}{n} + \frac{2}{n^2} - (b + \frac{1}{n})^2 = \frac{1}{n^2}$$

$$Var(B_{MLE:C}) = Var(B_{MLE} - \frac{1}{n}) = Var(B_{MLE}) = \frac{1}{n^2}$$

(5) $B_{MLE:C}$ is unbiased and has the same variance as B_{MLE}. Therefore $B_{MLE:C}$ is better than B_{MLE}. $B_{MLE:C}$ and B_{MOM} are both unbiased, but for $n > 1$, $B_{MLE:C}$ has smaller variance than B_{MOM}. Therefore $B_{MLE:C}$ is the best estimator.

Q2.

(1) $H_0 : \mu_1 = \mu_2$, $H_1 : \mu_1 \ne \mu_2$.
Equivalently, $H_0 : \mu_1 - \mu_2 = 0$, $H_1 : \mu_1 - \mu_2 \ne 0$.
$n = 25$, $m = 25$. With the equal variance assumption,

$$S_p^2 = 0.0195$$

$$TS = \frac{1.846 - 1.742}{\sqrt{0.0195\left(\frac{1}{25} + \frac{1}{25}\right)}} = 2.63$$

Since the test statistic, TS, is not in $(-t_{.995,48}, t_{.995,48}) = (-2.576, 2.576)$, reject H_0.

(2)$H_0 : \sigma_1^2 = \sigma_2^2$, $H_1 : \sigma_1^2 \ne \sigma_2^2$.
Equivalently, $H_0 : \frac{\sigma^2}{\sigma_2^2} = 1$, $H_1 : \frac{\sigma^2}{\sigma_2^2} \ne 1$.

$$
\begin{aligned}
TS &= \frac{S_1^2}{S_2^2} = .857 \\
F_{.005,24,24} &= 2.967 \\
F_{.995,24,24} &= \frac{1}{F_{.005,24,24}} = .337
\end{aligned}
$$

Since TS is in $(.337, 2.967)$, accept H_0.

(3) In part (2), we accepted $H_0 : \sigma_1^2 = \sigma_2^2$. Therefore, the assumption of equal variance in part (1) is valid, so the test in part(1) is valid.