Operations Research II, IEOR161

University of California, Berkeley

Midterm Exam II Suggested Solutions, 2009

1. (a) The states are:

   - $(2, 0)$: Both machines are operational
   - $(1, 0)$: One machine operational, one machine down but hasn't had any repairs done
   - $(1, 1)$: One machine operation, one machine has had one day's worth of repairs done
   - $(0, 1)$: Both machines down, one machine has been repaired for one day so far.

   (b) Let the states be in the order listed above. Then the transition probability matrix is

$$P = \begin{bmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & 1-p & p \\ 1-p & p & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

   (c)

$$\begin{aligned} \pi_{(2,0)} &= (1-p)\pi_{(2,0)} + (1-p)\pi_{(1,1)} \\ \pi_{(1,0)} &= p\pi_{(2,0)} + p\pi_{(1,1)} + \pi_{(0,1)} \\ \pi_{(1,1)} &= (1-p)\pi_{(1,0)} \\ \pi_{(0,1)} &= p\pi_{(1,0)} \\ \pi_{(2,0)} + \pi_{(1,0)} + \pi_{(1,1)} + \pi_{(0,1)} &= 1 \end{aligned}$$

2.

$$\pi_1 = \frac{3}{7},\ \pi_2 = \frac{1}{4},\ \pi_3 = \frac{9}{28}$$

3. Note that $\mathbb{E}\left[\sum_{j=1}^{N(T)} S_j\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{j=1}^{N(T)} S_j \middle| N(T)\right]\right]$. Now,

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{N(T)} S_j \middle| N(T)=n\right] &= \mathbb{E}\left[\sum_{j=1}^{n} S_j \middle| N(T)=n\right] \\
&= \sum_{j=1}^{n} \mathbb{E}\left[S_j | N(T)=n\right] \\
&= \sum_{j=1}^{n} \mathbb{E}[U] \qquad (1)\\
&= n\mathbb{E}[U] \qquad (2)\\
&= n\frac{T}{2} \qquad (3)
\end{aligned}$$

Where $U \sim \text{Uniform}(0,T)$. Line (1) follows from the fact that given we know that n arrivals occurred in $[0,T]$ according to a Poisson process, the arrival times S_j are distributed according to a uniform random variable on $[0,T]$. Now we uncondition on the value of $N(T)$:

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[\sum_{j=1}^{N(T)} S_j \middle| N(T)\right]\right] &= \mathbb{E}\left[N(T)\frac{T}{2}\right] \\
&= \frac{T}{2}\mathbb{E}[N(T)] \\
&= \frac{\lambda T^2}{2}
\end{aligned}$$

4. Given that these events have already occurred, if we measure time by the hour, the arrival time of each event is distributed according to a uniform random variable on $(0,1)$. The probability of *one* arrival coming between 12:15pm and 12:45pm, or the interval $\left(\frac{1}{4}, \frac{3}{4}\right)$, is $\frac{\frac{3}{4}-\frac{1}{4}}{1} = \frac{1}{2}$. Therefore, the probability of eight arrivals occurring in this time is $\left(\frac{1}{2}\right)^8$.