CS 170 Algorithms
Spring 2009 David Wagner

MT1 Soln

Midterm 1 solutions

Please do not read or discuss these solutions in the exam room while others are still taking the exam.

Problem 1. [True or false] (16 points)

Circle TRUE or FALSE. Do not justify your answers on this problem.

(a) $\boxed{\text{TRUE}}$ or FALSE: If $f(n) = (n+1)n/2$, then $f(n) \in O(n^2)$.

(b) $\boxed{\text{TRUE}}$ or FALSE: If $f(n) = (n+1)n/2$, then $f(n) \in \Theta(n^2)$.

(c) TRUE or $\boxed{\text{FALSE}}$: If $f(n) = (n+1)n/2$, then $f(n) \in \Theta(n^3)$.

(d) TRUE or $\boxed{\text{FALSE}}$: $n^{1.1} \in O(n(\lg n)^2)$.

(e) $\boxed{\text{TRUE}}$ or FALSE: It's possible to multiply two n-bit integers in $O(n^{1.9})$ time.

Comment: We saw that it was possible to multiple two n-bit integers in $O(n^{1.59})$ time, using a divide-and-conquer algorithm. So it's certainly possible to multiply them in $O(n^{1.9})$ time: just multiply using the divide-and-conquer algorithm, then execute no-op instructions to mark time until the total running time is $O(n^{1.9})$.

(f) $\boxed{\text{TRUE}}$ or FALSE: If vertices u, v are in the same strongly connected component of a directed graph G, then it is necessarily the case that v is reachable from u in G.

(g) TRUE or $\boxed{\text{FALSE}}$: If vertices u, v are *not* in the same strongly connected component of a directed graph G, then it is necessarily the case that v is *not* reachable from u in G.

Comment: It's possible that u, v are in two different SCCs and there is an edge from u to v.

(h) TRUE or $\boxed{\text{FALSE}}$: If we run breadth-first search on a directed graph G starting from vertex s, then depending on the graph, it might visit some vertices that a depth-first search starting from s would not visit. (Assume that we use exactly the breadth-first search algorithm specified in the book.)

Problem 2. [Recurrences] (16 points)

Write the solution to the following recurrences. Express your answer using $O(\cdot)$ notation. Do not justify your answers on this problem. Do not show your work.

(a) Solve the recurrence $F(n) = F(\lceil n/2 \rceil) + O(1)$.

Answer: $F(n) = O(\lg n)$.

(b) Solve the recurrence $F(n) = 4F(\lceil n/2 \rceil) + O(1)$.

Answer: $F(n) = O(n^2)$.

(c) Solve the recurrence $F(n) = 4F(\lceil n/2 \rceil) + O(n)$.

Answer: $F(n) = O(n^2)$.

(d) Solve the recurrence $F(n) = 4F(\lceil n/2 \rceil) + O(n^2)$.

Answer: $F(n) = O(n^2 \lg n)$.

Comment: All of these can be solved using the Master theorem, or by drawing a tree.

Problem 3. [Three-way mergesort] (18 points)

Alice suggests the following variant on mergesort: instead of splitting the list into two halves, we split it into three thirds. Then we recursively sort each third and merge them.

Mergesort3($A[0..n-1]$):
1. If $n \leq 1$, then return $A[0..n-1]$.
2. Let $k := \lceil n/3 \rceil$ and $m := \lceil 2n/3 \rceil$.
3. Return Merge3(Mergesort3($A[0..k-1]$), Mergesort3($A[k..m-1]$), Mergesort3($A[m..n-1]$)).

Merge3(L_0, L_1, L_2):
1. Return Merge(L_0, Merge(L_1, L_2)).

Assume that you have a subroutine Merge that merges two sorted lists of lengths ℓ, ℓ' in time $O(\ell + \ell')$. You may assume that n is a power of three, if you wish. Do not justify your answers on this problem. Do not show your work.

(a) What is the asymptotic running time for executing Merge3(L_0, L_1, L_2), if L_0, L_1, and L_2 are three sorted lists each of length $n/3$? Express your answer using $O(\cdot)$ notation.

Answer: $O(n)$.

Comment: The running time is $\frac{n}{3} + \frac{n}{3}$ for the call to Merge(L_1, L_2) and $\frac{n}{3} + \frac{2n}{3}$ for the outer call, for a total of $\frac{5n}{3}$, which is in $O(n)$.

(b) Let $T(n)$ denote the running time of Mergesort3 on an array of size n. Write a recurrence relation for $T(n)$.

Answer: $T(n) = 3T(n/3) + O(n)$.

Comment: There are three recursive calls to Mergesort3, each on a list of size $n/3$, followed by a call to Merge3, which takes $O(n)$ time by part (a).

(c) Solve the recurrence relation in part (b). Express your answer using $O(\cdot)$ notation.

Answer: $T(n) = O(n \lg n)$.

(d) Is the Mergesort3 algorithm asymptotically faster than insertion sort? Circle $\boxed{\text{YES}}$ or NO.

Comment: Insertion sort runs in $O(n^2)$ time, which is asymptotically slower.

(e) Is the Mergesort3 algorithm asymptotically faster than the ordinary mergesort? Circle YES or $\boxed{\text{NO}}$.

Comment: Ordinary mergesort runs in $O(n \lg n)$ time, which is asymptotically no faster or slower than Mergesort3.

Problem 4. [Algorithm design] (12 points)

Suppose we have t sorted arrays, each with n elements, and we want to merge them to get a single sorted array with tn elements. Your task is to fill in the blanks below to get an algorithm, ManyMerge, that solves this problem and has a running time of $O(nt \lg t)$.

You may assume that you are given an algorithm, Merge(L, L'), that merges two sorted lists of size ℓ, ℓ' into a single sorted list of size $\ell + \ell'$. You may assume that it runs in $O(\ell + \ell')$ time.

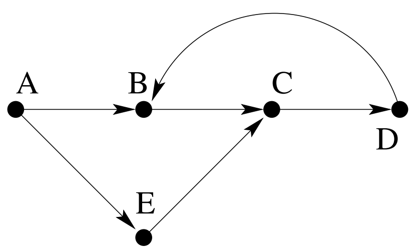
Fill in the empty boxes below, to get a correct algorithm whose running time is $O(nt \lg t)$.

ManyMerge($L_1[0..n-1], L_2[0..n-1], \ldots, L_t[0..n-1]$):
1. If $t = 1$, then return $\boxed{L_1}$.
2. If $t = 2$, then return Merge(L_1, L_2).
3. Set $L :=$ ManyMerge($\boxed{L_1, L_2, \ldots, L_{\lfloor t/2 \rfloor}}$).
4. Set $L' :=$ ManyMerge($\boxed{L_{\lfloor t/2 \rfloor + 1}, \ldots, L_{t-1}, L_t}$).
5. Return $\boxed{\text{Merge}(L, L')}$.

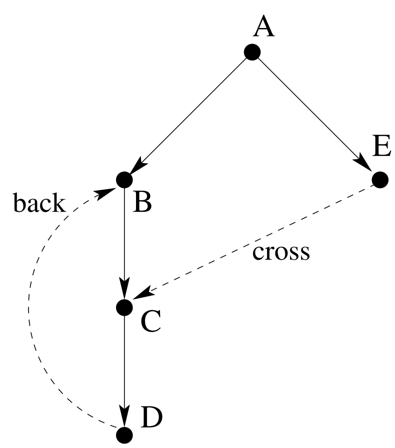
Comment: The running time of this solution satisfies the recurrence $T(t) = 2T(t/2) + O(nt)$: we recursively invoke ManyMerge twice, in each case with $t/2$ sorted lists of size n; and then we merge two lists of size $nt/2$, which takes $O(nt)$ time. The solution to this recurrence is $T(t) = O(nt \lg t)$, so the algorithm shown here does indeed achieve the necessary time bound. It is not hard to see why it is correct.

Problem 5. [Depth-first search] (16 points)

Run depth-first search on the directed graph below, starting at vertex A. Whenever there is a choice of the order to explore vertices, use alphabetical order (so A is chosen before B, and B before C, etc.).



(a) Draw the DFS tree that results, in the space provided below. Use solid lines for tree edges, and dotted lines for non-tree edges.



(b) Label each non-tree edge in the graph above with "forward", "back", or "cross", according to whether the edge is a forward edge, back edge, or cross edge.

(c) Can the above graph be topologically sorted? Circle YES or $\boxed{\text{NO}}$. Do not justify your answer.

(d) How many strongly connected components does this graph have? Do not justify your answer.

Answer: 3.

Comment: The three strongly connected components are: $\{A\}$, $\{E\}$, and $\{B,C,D\}$.

Problem 6. [Short answer] (22 points)

Answer each question below *concisely* (one short sentence or a number should suffice). Do not justify your answer. Do not show your work.

(a) Suppose we are given a directed graph $G = (V,E)$ represented in adjacency list format, and we want to test whether G is a dag or not, using a method that is as asymptotically efficient as possible. In a sentence, what approach would you use?

Answer 1: Use DFS, check for back-edges.

Answer 2: Decompose into strongly connected components, check for a SCC with more than one vertex.

Comment: There is a cycle (and hence G fails to be a dag) if and only if DFS finds a back edge. There is a cycle (and hence G fails to be a dag) if and only if there is a strongly connected component with more than one vertex.

(b) What's the running time of your solution in (a), using $O(\cdot)$ notation?

Answer: $O(|V|+|E|)$.

(c) Let $G = (V,E)$ be a directed graph with $|V| = 1000$ vertices, $|E| = 5000$ edges, and 700 strongly connected components. How many vertices does the metagraph have?

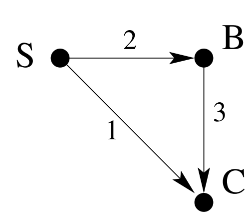
Answer: 700.

Comment: Each vertex in the metagraph corresponds to a strongly connected component in G, so the number of vertices in the metagraph is the same as the number of SCCs in G.

(d) Let $G = (V,E)$ be a dag, where each edge is annotated with some positive length. Let s be a source vertex in G. Suppose we run Dijkstra's algorithm to compute the distance from s to each vertex $v \in V$, and then order the vertices in increasing order of their distance from s. Are we guaranteed that this is a valid topological sort of G? Circle YES or $\boxed{\text{NO}}$.

(e) Justify your answer to part (d) as follows: If you circled YES, then give one sentence that explains the main idea in a proof of this fact. If you circled NO, then give a small counterexample (a graph with at most 4 vertices) that disproves it.

Answer:



Comment: Sorting by increasing distance gives S,C,B; but B must precede C in any valid topological sort.

(f) Suppose we run Dijkstra's algorithm on a graph with n vertices and $O(n \lg n)$ edges. Assume the graph is represented in adjacency list representation. What's the asymptotic running time of Dijkstra's algorithm, in this case, if we use a binary heap for our priority queue? Express your answer as a function of n, and use $O(\cdot)$ notation.

Answer: $O(n(\lg n)^2)$.

Comment: $|V| = n$, $|E| = O(n \lg n)$, and Dijkstra's runs in $O((|V| + |E|) \lg |V|)$ time, which is $O((n + O(n \lg n)) \lg n) = O((n \lg n) \times \lg n) = O(n(\lg n)^2)$.