Student name:______________________________

Problem 1. (20+10 = 30 points)

The radioactivity of a material at time t is given by the formula

$$R_t = R_0 \exp(-At)$$

where R_0 is the radioactivity at time $t = 0$ and A is the rate of percent decay per unit time. Suppose R_0 and A are random variables with the second moments listed below:

Variable	Mean	Standard deviation	Correlation coefficients	
			R_0	A
R_0	100	30	1	0.3
A	0.05	0.02	0.3	1

a) For $t = 10$, using first-order approximations, determine the mean and standard deviation of R_t and the correlation coefficient between R_t and A.

b) Determine the relative importance of the random variables R_0 and A in contributing to the variability of R_t.

a) $\mu_{R_t} \cong 100 \times \exp(-0.05 \times 10) = 60.653$ *Ans.*

$$\left(\frac{\partial R_t}{\partial R_0}\right)_{x=M} = \left[\exp(-At)\right]_{x=M} = \exp(-0.05 \times 10) = 0.607$$

$$\left(\frac{\partial R_t}{\partial A}\right)_{x=M} = \left[tR_0 \exp(-At)\right]_{x=M} = -10 \times 100 \times \exp(-0.05 \times 10) = -606.530$$

$$\sigma^2_{R_t} \cong (0.607)^2(30)^2 + (-606.530)^2(0.02)^2 + 2(0.607)(-606.530)(0.3)(30)(0.02)$$
$$= 331.091 + 147.152 - 132.437 = 345.806$$

$\sigma_{R_t} = 18.6$ *Ans.*

$$\text{Cov}[R_t, A] \cong (-606.530)(1)(0.02)^2 + (0.607)(1)(0.3)(30)(0.02) = -0.133$$

$$\rho_{R_t A} \cong \frac{-0.133}{(18.6)(0.02)} = -0.358 \quad \textit{Ans.}$$

b) $\text{Imp}(R_t) = |0.607| \times 30 = 18.196, \quad \text{Imp}(A) = |-606.530| \times 0.02 = 12.131$

$\text{Imp}(R_t) > \text{Imp}(A)$ *Ans.*

Problem 2. (10+20 = 30 points)

The natural period T of a structure is given in terms of its circular frequency ω by the relation

$$T = \frac{2\pi}{\omega}$$

Suppose ω has the lognormal distribution with a mean of $\mu_\omega = 10$ rad/s and standard deviation $\sigma_\omega = 3$ rad/s.

a) Determine the parameters λ and ζ of the lognormal distribution of ω.

b) Derive an expression for the distribution of T. Identify this distribution (give its name) and determine the values of its parameters.

a) $\zeta = \sqrt{\ln(1+0.3^2)} = 0.294$ *Ans.*

$\lambda = \ln(10) - 0.5(0.294)^2 = 2.259$ *Ans.*

b) inverse relation is

$$\omega = \frac{2\pi}{t} \qquad \frac{d\omega}{dt} = -\frac{2\pi}{t^2}$$

$$f_T(t) = \frac{1}{\sqrt{2\pi}\zeta\omega}\exp\left[-\frac{1}{2}\left(\frac{\ln(\omega)-\lambda}{\zeta}\right)^2\right]\left|-\frac{2\pi}{t^2}\right|$$

$$= \frac{t}{\sqrt{2\pi}\zeta(2\pi)}\exp\left[-\frac{1}{2}\left(\frac{\ln(2\pi/t)-\lambda}{\zeta}\right)^2\right]\frac{2\pi}{t^2}$$

$$= \frac{1}{\sqrt{2\pi}\zeta t}\exp\left[-\frac{1}{2}\left(\frac{-\ln(t)+\ln(2\pi)-\lambda}{\zeta}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\zeta t}\exp\left[-\frac{1}{2}\left(\frac{\ln(t)-[\ln(2\pi)-\lambda]}{\zeta}\right)^2\right] \qquad 0 < t$$

T has the lognormal distribution with parameters: *Ans.*

$\lambda_T = \ln(2\pi) - \lambda = -0.421$

$\zeta_T = \zeta = 0.294$ *Ans.*

Student name:________________________________

Problem 3. (8+8+8+8+8 = 40 points)

The occurrence of damaging earthquakes in a region is modeled by the Poisson process, whereby the number of earthquakes to occur during an interval $(0,t)$ is described by the probability mass function

$$p_N(n) = \frac{(\nu t)^n \exp(-\nu t)}{n!} \qquad n = 0,1,2,\ldots$$

where $\nu = 0.05$ is the mean rate per year.

a) Determine the mean and standard deviation of the number of earthquakes to occur in 50 years.

b) What is the probability that two earthquakes will occur in 10 years?

c) What is the probability that at least two earthquakes will occur in 10 years?

d) What is the probability that the first earthquake will occur in less than 10 years?

e) If only 20% of such earthquakes result in loss of life, what is the probability that lives will be lost due to earthquakes in 10 years?

---

a) $\mu_N = 0.05 \times 50 = 2.5$ *Ans.*

$\sigma_N = \sqrt{0.05 \times 50} = 1.581$ *Ans.*

b) $$\text{P(two earthquakes in 10 years)} = \frac{(0.05 \times 10)^2 \exp(-0.05 \times 10)}{2!} = 0.0758$$ *Ans.*

c) P(at least two earthquakes in 10 years)

$$= 1 - \frac{(0.05 \times 10)^0 \exp(-0.05 \times 10)}{0!} - \frac{(0.05 \times 10)^1 \exp(-0.05 \times 10)}{1!}$$

$= 1 - 0.607 - 0.303 = 0.0902$ *Ans.*

d) Time to the first occurrence in the Poisson process has the exponential distribution. Thus,

P(first earthquake before 10 years) $= F_T(10) = 1 - \exp(-0.05 \times 10) = 0.393$ *Ans.*

e) Earthquakes causing loss of life are Poisson with mean rate $0.20 \times 0.05 = 0.01$.

$$\text{P(lives lost in 10 years)} = 1 - \frac{(0.01 \times 10)^0 \exp(-0.01 \times 10)}{0!} = 0.0951$$ *Ans.*